RESEARCH STATEMENT

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My research fits in the theory of noncommutative rings and their modules and in their interactions with lattices. Principally, I work in different generalizations of properties of rings to modules. In my dissertation [13], I generalized a famous theorem of Alfred Goldie to modules. I defined *semiprime Goldie modules* and characterized them in terms of uniform dimension and M-singularity as A. Goldie did for semiprime Goldie rings [8]. After I finished my Ph.D. I kept studying the Goldie modules [6, 7] and the modules satisfying *ACC on M-annihilators* [5]. In the theory of Goldie modules there are some questions I want to solve. It can be seen that under some projectivity assumptions if a module M is Goldie, then its endomorphism ring is right Goldie [3, Lemma 5.3]. But if a module has a right (semiprime) Goldie endomorphism ring, the module need not to be a (semiprime) Goldie module. I would like to find the conditions which characterize those modules whose endomorphism rings are (semiprime) right Goldie rings. On the other hand, since we are working with modules, it is always interesting to study the direct sums. In [7, Lemma 2.6] was proved that the finite direct sum of copies of a semiprime Goldie module is a semiprime Goldie module. It remains open the general case, that is: Is a finite direct sum $\bigoplus_{i=1}^{n} M_i$ with M_i a (semiprime) Goldie module, a (semiprime) Goldie module? Also in [7, Example 2.10], the semiprime Goldie $\mathbb{Z}$-modules were described. So, it is possible that we can describe the semiprime Goldie R-modules for more general rings, for example Dedekind domains or Noetherian rings. Finally, in this subject, we can also consider the class of all (semiprime) Goldie modules for a fixed ring R and study its closure properties.

I want to study the consequences of having a definition of a Goldie module. I already studied that the intersection of all prime submodules is a nilpotent submodule (as in Goldie rings) [3]. So, it is possible to study the *finite reduced rank* as in [2] and to give a module theoretic version of the *Small's Theorem* regarding orders in Artinian rings [4]. This opens the door to study all the consequences of the Small's Theorem but in the module theoretic realm.

Another immediate consequence of having the concept of a Goldie module is to study the analogous of hereditary prime Goldie rings. These rings are more general than hereditary prime Noetherian rings which emerge as a noncommutative counterpart of the Dedekind rings. In [9], we defined Σ-*Rickart modules* as a notion which generalizes hereditary rings. Therefore, it seems natural to study Σ-*Rickart Goldie modules.*

The notion of *Rickart module* was introduced in [11] as a generalization of the rings with the same name. A module M is N-Rickart if $\operatorname{Ker}\varphi$ is a direct summand

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of M for all $\varphi : M \to N$. Then we can define a class of modules $\mathcal{R}(M) = \{N \mid M \text{ is } N\text{-Rickart}\}$ and study the closure properties of this class. I want to know if the class $\mathcal{R}(M)$ fits in some known kind of classes of modules, such as natural classes, pretorsion classes, hereditary torsion classes, etc. and when a class of modules can be realized as $\mathcal{R}(M)$ for some module M. Also, this can be done for the dual concept, that is, *dual-Rickar modules.* It is known that the study of classes of modules can give us information about the categoy or the base ring.

It is possible to carry many other notions of rings to modules. I am interested in study those modules whose maximal submodules are fully invariant, called *quasi-duo modules.* This is because these modules have appeared in some of my works.

As I mentioned at the beginning, I also study the relations between rings, modules and lattices. It is possible to define a *product of submodules* of a given R-module. This product is not associative nor distributive over sums, in general. There are some conditions which make this product associative and distributive, that is, the lattice of submodules of a module is a *quantale.* For example, the lattice of submodules of any projective module is always a quantale. Many of the generalizations from rings to modules (that I have made) have hypotheses to make the lattice of submodules a quantale. So, it is important to know what are those modules whose lattice of submodules is a quantale because those are the good modules to make the generalizations. I would like to find the necessary and sufficient conditions to know when the lattice of submodules of a module is a quantale.

Since it is possible to define the concept of prime submodule, it is possible to define a prime spectrum of a module [12, 14]. It can be seen that many classical results on the prime spectrum of a ring can be translated to modules. In fact, those results can be proved in the context of lattices in the structures that we call *quasi-quantales.* So, at some point it should be possible to define a sheaf in the prime spectrum of a quasi-quantale and to find a connection of the quasi-quantale with sections of that sheaf. Latter this can be applied to modules.

In some cases, the language of lattices can be the right approach to an algebraic problem. Many definitions in modules concerns only the properties of the lattice of submodules. For example, Noetherian module, Artinian module, finitely generated and so on. Also, there are other definitions which concerns the lattice of submodules and the application of the morphisms on the submodules. For example, Baer and Rickart modules. In [1] were defined *linear morphisms* between complete modular lattices as a reflection of the behavior of an R-morphism in the lattice of submodules. So, the category of linear modular lattices seems to be a good place to study and extend properties of modules. In [15], we saw that the category of linear modular lattices was the right place to study the concepts of Rickart and Baer modules and their duals. In that way, it is possible to extend other concepts to this category such as endoregular modules [10]. One of the advantages of working in this context is that the dual results come, most of the time, easily.

As it is known, in the category of all R-modules, there are some subclasses of modules defined by closure properties which are very interesting and useful. Perhaps the most common example of these classes are the *hereditary torsion classes* or *localizing classes.* H. Simmons [18] developed a theory in lattices inspired in classes of modules. One of the motivations is that given a ring R, the localizing classes in R-Mod are in bijective correspondence with some operators called nuclei, on the lattice of left ideals of R. Given a complete modular lattice $\mathcal{L}$, we consider the set of all intervals in $\mathcal{L}$, that is, $I(\mathcal{L}) = \{[a, b] \mid a \leq b \in \mathcal{L}\}$. We can think of $I(\mathcal{L})$ as the category of modules and subsets of $I(\mathcal{L})$ as classes of modules. The subset analogous to the localizing classes are called *division sets* and it is proved that, theses division sets are in one-to-one correspondence with the nuclei on $\mathcal{L}$. In this theory, my colleagues and I have been able to give a counterpart of the theory of hereditary torsion theories and dimensions related to them [16, 17]. As a current project we are studying the analogous, in this context of the Goldie's torsion theory. The Goldie's torsion theory is that given by the singular modules. This torsion theory is important because it is related with the maximal ring of quotients and the classical ring of quotients of a ring, also in abelian groups, this torsion theory coincides with the class of torsion groups. In the context of torsion theories we can find other special classes, for example torsion torsion-free classes (TTF) and spectral torsion theories. My colleagues and I are studying these two concepts in the context of subsets of intervals in a lattice. One of the goals of this development is to give a detailed theory of localization on lattices involving these subsets of intervals as in the case of R-Mod.

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