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COMPLEMENTARY NOTES TO THE BOOK “LECTURES ON ALGEBRAIC QUANTUM GROUPS”*

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Abstract.

1. Hopf Algebras

Let us fix a fiel k.

Given two vector spaces (over k) A anb B, we will denote by $A \otimes B$ the tensor product over k.

Definition 1.1. Let A and B be vector spaces. The *twist map*

$$\tau : A \otimes B \longrightarrow B \otimes A$$

is the k-morphism sending $\tau(a \otimes b) = b \otimes a$.

Definition 1.2. An *algebra over* k or a k*-algebra* is a vector space A equipped with k-morphisms $\mu : A \otimes A \longrightarrow A$ and $u : k \longrightarrow A$ called the multiplication and the unit respectively, such that the following diagrams commute:

$$\begin{array}{ccccc} k \otimes A & \xrightarrow{u \otimes I_A} & A \otimes A & \xleftarrow{I_A \otimes u} & A \otimes k \\ & \searrow^{\cong} & \downarrow{\mu} & \swarrow^{\cong} & \\ & & A & & \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes I_A} & A \otimes A \\ \downarrow{I_A \otimes \mu} & & \downarrow{\mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

A is commutative if the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ & \searrow^{\mu} \quad \swarrow^{\mu} & \\ & A & \end{array}$$

Example 1.3.
(i) k is a commutative k-algebra with multiplication $\mu(a \otimes b) = ab$ and unit $u(1) = 1$.
(ii) $k[x]$ the polynomial ring with coefficients in k.
(iii) $M_n(k)$ the n by n matrices with coefficients in k.
(iv) Tensor product of two algebras. Let (A, μ, u) and (A', μ', u) be two k-algebras. Then $A \otimes A'$ is a k-algebera with the following multiplication and unit:

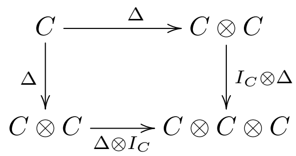
$$\mu_\otimes : (A \otimes A') \otimes (A \otimes A') \xrightarrow{I_A \otimes \tau \otimes I_{A'}} (A \otimes A) \otimes (A' \otimes A') \xrightarrow{\mu \otimes \mu'} A \otimes A'$$

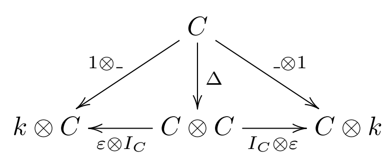
$$u_\otimes(1) = u(1) \otimes u'(1)$$

*Written by K. Brown and K. Goodearl [1].

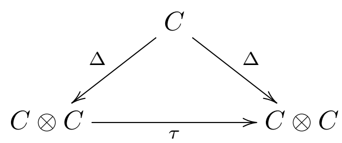
Now, we can dualize the diagrams in the definition of k-algebraa and we get a k-coalgebra:

Definition 1.4. A *coalgebra over* k or a k*-coalgebra* is a vector space C together k-morphisms $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$ called the comultiplication and the counit respectively, such that the following diagrams commute:





C is cocommutative if the following diagram commutes:



Sweedler's Notation

For an element $c \in C$, we will write $\Delta(c) = \sum c_1 \otimes c_2 \in C \otimes C$ where c_1 and c_2 refer to variables elements of C, not uniquely, determinated. The subscripts 1 and 2 indicate the position of these elements in the tensor product. With this notation the coassociativity of Δ is expressed by:

$$
\begin{aligned}
(I_C \otimes \Delta)\Delta(c) &= (I_C \otimes \Delta)(\sum c_1 \otimes c_2) \\
&= \sum c_1 \otimes \Delta(c_2) \\
&= \sum c_1 \otimes (c_{21} \otimes c_{22}) \\
&= \sum (c_{11} \otimes c_{12}) \otimes c_2 \\
&= \sum \Delta(c_1) \otimes c_2 \\
(\Delta \otimes I_C)\Delta(c) &= (\Delta \otimes I_C)(\sum c_1 \otimes c_2)
\end{aligned} \tag{1.1}
$$

For the counit:

$$
\begin{aligned}
(\varepsilon \otimes I_C)\Delta(c) &= (\varepsilon \otimes I_C)(\sum c_1 \otimes c_2) \\
&= \sum \varepsilon(c_1) \otimes c_2 \\
&= 1 \otimes \sum \varepsilon(c_1)c_2 \\
&= 1 \otimes c
\end{aligned}
$$

This implies that,

$$c = \sum \varepsilon(c_1)c_2 \tag{1.2}$$

Analogously,

$$c = \sum c_1\varepsilon(c_2) \tag{1.3}$$

The cocommutativity is equivalent to

$$\sum c_1 \otimes c_2 = \sum c_2 \otimes c_1 \tag{1.4}$$

Example 1.5. (i) k is a coalgebra with comultiplication $\Delta(a) = 1 \otimes a$ and counit $\varepsilon = I_k$.

(ii) Let F be a vector space with basis $\{f_\lambda\}_{\lambda \in \Lambda}$. Then there exist an unique k-morphism $\Delta : F \longrightarrow F \otimes F$ such that $\Delta(f_\lambda) = f_\lambda \otimes f_\lambda$ for all $\lambda \in \Lambda$; and a unique k-morphism $\varepsilon : F \longrightarrow k$ such that $\varepsilon(f_\lambda) = 1$ for all $\lambda \in \Lambda$.

Coassociativity

$$(I_F \otimes \Delta)\Delta(f_\lambda) = (I_F \otimes \Delta)(f_\lambda \otimes f_\lambda) = f_\lambda \otimes \Delta(f_\lambda) = f_\lambda \otimes f_\lambda \otimes f_\lambda$$

$$(\Delta \otimes I_F)\Delta(f_\lambda) = (\Delta \otimes I_F)(f_\lambda \otimes f_\lambda) = \Delta(f_\lambda) \otimes f_\lambda = f_\lambda \otimes f_\lambda \otimes f_\lambda$$

Counit For the counit we have

$$f_\lambda = 1 f_\lambda = \varepsilon(f_\lambda) f_\lambda$$

(iii) Let G be a group. Consider the group ring $k[G]$. Then, we can give two coalgebra's structures to $k[G]$ as follows: The first structure is given by last example. That is, $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. For the other structure, let $e \in G$ the unit of G. We define

$$\Delta : k[G] \longrightarrow k[G] \otimes k[G]$$

as follows:

$$\Delta(g) = \begin{cases} e \otimes e & \text{if } g = e \\ g \otimes e + e \otimes g & \text{if } g \neq e \end{cases}$$

and counit,

$$\varepsilon : k[G] \longrightarrow k$$

as:

$$\varepsilon(g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

(iv) Consider the polynomial ring $k[x]$. We can give two coalgebra structures to $k[x]$. The first is given by the example (ii), with the canonical basis $\{1, x, x^2, x^3, \dots\}$. The second structure is as follows:

Comultiplication:

$$\Delta : k[x] \longrightarrow k[x] \otimes k[x]$$

defined as

$$\Delta(x^i) = (x \otimes 1 + 1 \otimes x)^i$$

Counit:

$$\varepsilon : k[x] \longrightarrow k$$

defined as

$$\varepsilon(x^i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}$$

Let us check the coassociativity,

$$\begin{aligned}(1\otimes\Delta)\Delta(x^i) &= (1\otimes\Delta)(x\otimes 1+1\otimes x)^i\\ &= (1\otimes\Delta)\sum_{j=0}^{i}k_j(x^j\otimes x^{i-j})\\ &= \sum_{j=0}^{i}k_i(x^j\otimes\Delta(x^{i-j}))\\ &= \sum_{j=0}^{i}k_i(x^j\otimes\sum_{\ell=0}^{i-j}a_\ell(x^\ell\otimes x^{i-j-\ell}))\\ &= \sum_{j=0}^{i}\sum_{\ell=0}^{i-j}k_ia_\ell(x^j\otimes x^\ell\otimes x^{i-j-\ell})\end{aligned}$$

For some elements $k_j, a_\ell \in k$. On the other hand,

$$\begin{aligned}(\Delta\otimes 1)\Delta(x^i) &= (\Delta\otimes 1)(x\otimes 1+1\otimes x)^i\\ &= (\Delta\otimes 1)\sum_{j=0}^{i}k_j(x^j\otimes x^{i-j})\\ &= \sum_{j=0}^{i}k_i(\Delta(x^j)\otimes x^{i-j})\\ &= \sum_{j=0}^{i}k_i(\sum_{\ell=0}^{j}a_\ell(x^\ell\otimes x^{j-\ell})\otimes x^{i-j})\\ &= \sum_{j=0}^{i}\sum_{\ell=0}^{j}k_ia_\ell(x^\ell\otimes x^{j-\ell}\otimes x^{i-j})\end{aligned}$$

It follows that $(\Delta\otimes 1)\Delta(x^i) = (1\otimes\Delta)\Delta(x^i)$. Now,

$$\begin{aligned}(\varepsilon\otimes 1)\Delta(x^i) &= (\varepsilon\otimes 1)\sum_{j=0}^{i}k_j(x^j\otimes x^{i-j})\\ &= \sum_{j=0}^{i}k_j(\varepsilon(x^j)\otimes x^{i-j})\\ &= 1\otimes x^i\end{aligned}$$

Analogously, $(1\otimes\varepsilon)\Delta(x^i) = x^i\otimes 1$. Thus $(k[x],\Delta,\varepsilon)$ is a coalgebra.

(v) Consider the matrix ring $\mathrm{Mat}_n(k)$ with canonical basis $\{e_{ij}\}$. Define

$$\Delta : \mathrm{Mat}_n(k) \longrightarrow \mathrm{Mat}_n(k)\otimes\mathrm{Mat}_n(k)$$

as follows

$$\Delta(e_{ij}) = \sum_{\ell} e_{i\ell}\otimes e_{\ell j}$$

And

$$\varepsilon : \mathrm{Mat}_n(k) \longrightarrow k$$

as

$$\varepsilon(e_{ij}) = \delta_{ij}$$

We will call this coalgebra the $n \times n$ matrix coalgebra over k and we will denote it by $\mathrm{Mat}_n^c(k)$. We can identify this coalgebra with the ring of polynomial functions on the space of $n \times n$ matrices over k

$$\mathcal{O}(\mathrm{Mat}_n(k)) = k[X_{ij} \mid 1 \leq i, j \leq n]$$

sending $e_{ij} \longmapsto X_{ij}$.

(vi) The tensor product of two coalgebras. Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be two coalgebras. Then the tensor product $C \otimes C'$ is a coalgebra with comultiplication and unit:

$$\Delta_{\otimes} : C \otimes C' \xrightarrow{\Delta\otimes\Delta} C \otimes C \otimes C' \otimes C' \xrightarrow{I_C \otimes \tau \otimes I_{C'}} C \otimes C' \otimes C \otimes C'$$

$$\varepsilon_{\otimes} : C \otimes C' \xrightarrow{\varepsilon\otimes\varepsilon} k$$

Convolution Product

Example 1.6. Consider any coalgebra (C, Δ, ε). Let C^* denote the dual of C as vector space, that is $C^* = \mathrm{Hom}_k(C, k)$. Then C^* is a k-algebra with product given by the transpose of Δ, that is, for $f, g \in C^*$, $(f * g)(c) = \sum f(c_1)g(c_2)$ where $\Delta(c) = \sum c_1 \otimes c_2$.

Proposition 1.7. *Let (A, μ, u) be an k-algebra and let (C, Δ, ε) be a k-coalgebra. Then $\mathrm{Hom}_k(C, A)$ is a k-algebra with multiplication and unit as follows: Let $f, g \in \mathrm{Hom}_k(C, A)$*

$$* : \mathrm{Hom}_k(C, A) \otimes \mathrm{Hom}_k(C, A) \longrightarrow \mathrm{Hom}_k(C, A)$$

$$f * g = \mu(f \otimes g)\Delta$$

The unit is given by the composition $u\varepsilon$.

In the Sweedler's notation

$$(f * g)(c) = \sum f(c_1)g(c_2)$$

$$u\varepsilon(c) = \varepsilon(c)1_A$$

This product is called the convolution product.

Proof. Associativity.

$$\begin{aligned} f * (g * h) &= \mu(f \otimes (g * h))\Delta \\ &= \mu(f \otimes \mu(g \otimes h)\Delta)\Delta \\ &= \mu\left((I_A \otimes \mu)(f \otimes (g \otimes h))(I_C \otimes \Delta)\right)\Delta \\ &= \mu((\mu \otimes I_A)((f \otimes g) \otimes h)(\Delta \otimes I_C))\Delta \\ &= \mu((\mu(f \otimes g) \otimes h)(\Delta \otimes I_C))\Delta \\ &= \mu(\mu(f \otimes g)\Delta \otimes h)\Delta \\ &= (f * g) * h \end{aligned}$$

Now, let us check that $u\varepsilon$ is the unit.

$$\begin{aligned}(f * u\varepsilon)(c) &= \sum f(c_1)u\varepsilon(c_2)\\ &= \sum f(c_1)\varepsilon(c_2)1_A\\ &= \sum f(c_1\varepsilon(c_2))\\ &= f(\sum c_1\varepsilon(c_2))\\ &= f(c)\end{aligned}$$

Last equality follows from (1.2). Analogously, $u\varepsilon * f = f$. □

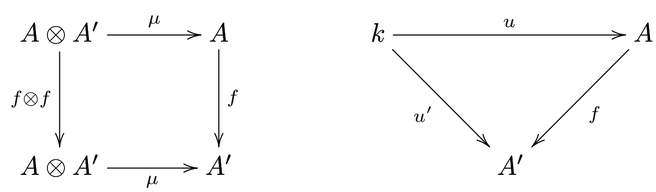
The dual of Proposition 1.7 fails since V is an infinite dimensional vector space, $V^* \otimes V^*$ is a proper subspace of $(V \otimes V)^*$, so that the dual of the multiplication map on an infinite dimensional algebra A need not take all values in $A^* \otimes A^*$.

Definition 1.8. Let A be a k-algebra. The finite dual of Hopf dual of A is the set

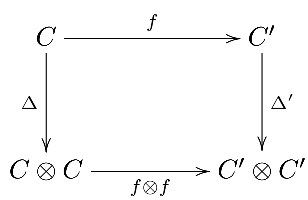
$$A^\circ = \{f \in A^* \mid f(I) = 0 \text{ for some ideal } I \text{ of } A \text{ with } dim_k(A/I) < \infty\}$$

Proposition 1.9. *Let (A, μ, u) be a k-algebra. Then A° is a coalgebra with comultiplication $\Delta = \mu^*$ and counit $\varepsilon = u^*$.*

Definition 1.10. Let (A, μ, u) and (A', μ', u') be two algebras. A k-morphism $f : A \longrightarrow A'$ is an algebra morphism if the following diagrams commute:



Definition 1.11. Let (C, Δ, ε) and $(C', \Delta', \varepsilon')$ be two coalgebras. A k-morphism $f : C \longrightarrow C'$ is a coalgebra morphism if the following diagrams commute:



Definition 1.12. Let (C, Δ, ε) be a coalgebra. A subspace I of C is a coideal if $\Delta(I) \subseteq C \otimes I + I \otimes C$ and $\varepsilon(I) = 0$.

Proposition 1.13. *Let I be a coideal of a coalgebra (C, Δ, ε). Then*

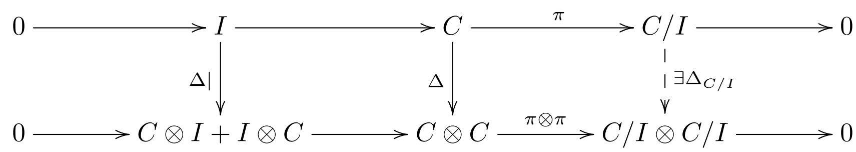
(a) *C/I is a coalgebra.*
(b) *The canonical projection $\pi : C \longrightarrow C/I$ is a coalgebra morphism.*

Proof. *(a)* Consider $\pi \otimes \pi : C \otimes C \longrightarrow C/I \otimes C/I$. Then $Ker(\pi \otimes \pi) = C \otimes I + I \otimes C$. This implies that

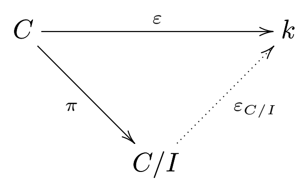
$$\frac{C \otimes C}{C \otimes I + I \otimes C} \cong \frac{C}{I} \otimes \frac{C}{I}$$

□

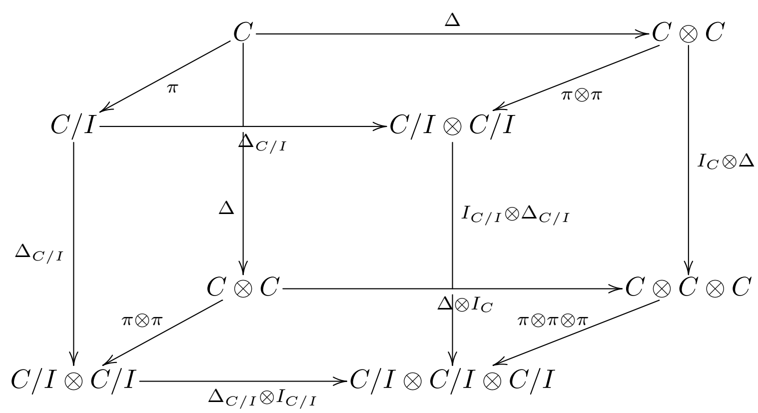
Hence the comultiplication $\Delta_{C/I}$ on C/I is induced by the following diagram:



The counit is the map induced by ε,



For the coassociativity, we have to see that front face of the next cube commutes:



The back face is the coassociativity of Δ. Notice that by the definition of $\Delta_{C/I}$ the top, bottom, right and left faces commute. Hence

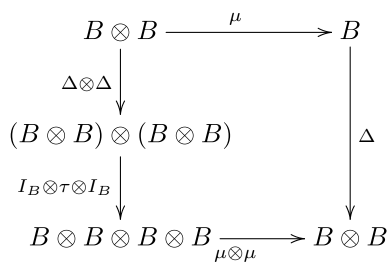
$$\begin{aligned}(I_{C/I} \otimes \Delta_{C/I})\Delta_{C/I}\pi &= (I_{C/I} \otimes \Delta_{C/I})(\pi \otimes \pi)\Delta \\ &= (\pi \otimes \pi \otimes \pi)(I_C \otimes \Delta) \\ &= (\Delta_{C/I} \otimes I)(\pi \otimes \pi)\Delta \\ &= (\Delta_{C/I} \otimes I_{C/I})\Delta_{C/I}\pi\end{aligned}$$

Since π is surjective, $(I_{C/I} \otimes \Delta_{C/I})\Delta_{C/I} = (\Delta_{C/I} \otimes I_{C/I})\Delta_{C/I}$.
It is left to the reader to see that $\varepsilon_{C/I}$ is a counit.
(b) It is clear for the construction of $\Delta_{C/I}$.

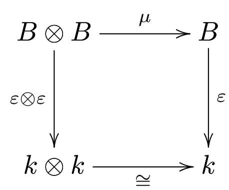
Definition 1.14. A k-Bialgebra is a k-vector space B equipped with linear maps $\mu, u, \Delta, \varepsilon$ such that (B, μ, u) is a k-algebra, (B, Δ, ε) is a k-coalgebra and either:

- Δ and ε are algebra morphisms, or
- μ and u are coalgebra morphisms.

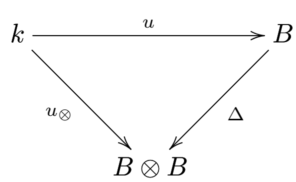
In fact, these two last conditions are equivalent. For, suppose Δ and ε are algebra morphisms. Consider the following diagram,



If we recall the tensor algebra (Example 1.3(iv)) and the tensor coalgebra (Example 1.5(vi)) then we can see that if Δ is an algebra morphism then the diagram commutes. And, if ε is an algebra morphism then the following diagram commutes



Thus μ is a coalgebra morphism. To see that u is a coalgebra morphism, since ε is an algebra morphism then $\varepsilon u = I_k$ and since Δ is an algebra morphism this diagram



commutes. So u is a coalgebra morphism. The other implication follows with the same diagrams.

Example 1.15. (i) Consider the coalgebra $k[G]$ with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon(g) = 1$ (Example 1.5(iii)). Then $k[G]$ is a bialgebra. Let see that Δ is an algebra morphism.

$$\begin{aligned}\mu_\otimes(\Delta\otimes\Delta)(g\otimes h) &= (\mu\otimes\mu)(I\otimes\tau\otimes I)(\Delta\otimes\Delta)(g\otimes h)\\ &= (\mu\otimes\mu)(I\otimes\tau\otimes I)(g\otimes g\otimes h\otimes h)\\ &= \mu\otimes\mu(g\otimes h\otimes g\otimes h)\\ &= gh\otimes gh\\ &= \Delta(gh)\\ &= \Delta\mu(g\otimes h)\end{aligned}$$

And

$$\begin{aligned}\Delta u(1) &= \Delta(1)\\ &= 1\otimes 1\\ &= u_\otimes(1)\end{aligned}$$

Thus, Δ is an algebra morphism. It is clear that ε is an algebra morphism.

(ii) Consider the coalgebra $\mathcal{O}(\mathrm{Mat}_n(k))$ presented in Example 1.5(v) and consider the isomorphism

$$\mathcal{O}(\mathrm{Mat}_n(k)) \otimes \mathcal{O}(\mathrm{Mat}_n(k)) \xrightarrow[\cong]{\varphi} \mathcal{O}(\mathrm{Mat}_n(k) \times \mathrm{Mat}_n(k))$$

where $\varphi(f \otimes g)(a, b) = f(a)g(b)$. Let $m : \mathrm{Mat}_n(k) \times \mathrm{Mat}_n(k) \longrightarrow \mathrm{Mat}_n(k)$ denote the multiplication of matrices. Then the comultiplication in $\mathcal{O}(\mathrm{Mat}_n(k))$ is given by $\Delta(f) = \varphi^{-1}(fm)$. Given $f, g \in \mathcal{O}(\mathrm{Mat}_n(k))$ denote the product of polynomials as $f \cdot g$. Then

$$\Delta(f \cdot g) = \varphi^{-1}((f \cdot g)m) = \varphi^{-1}(fm \cdot gm) = \varphi^{-1}(fm)\varphi(gm) = \Delta(fm)\Delta(gm).$$

It is clear that $\Delta(1) = 1 \otimes 1$. Hence Δ is an algebra morphism. On the other hand, the counit of this coalgebra is $\varepsilon(f) = f(I_n)$ where I_n is de identity matrix, so itis clear that ε is an algebra morphism. Thus $\mathcal{O}(\mathrm{Mat}_n(k))$ is a bialgebra.

(iii) The *quantum plane*

$$\mathcal{O}_q(k^2) = k\langle x, y \mid xy = qyx\rangle$$

We have that $\mathcal{O}_q(k^2) \cong k[x][y; \tau]$ (an skew polynomial ring) where $\tau : k[x] \longrightarrow k[x]$ is the automorphism given by $\tau(f(x)) = f(q^{-1}x)$ and so $\{x^i y^j \mid i, j \geq 0\}$ is a basis for this algebra. $\mathcal{O}_q(k^2)$ is a bialgebra with comultiplication given by

$$\Delta(x) = x \otimes x$$

$$\Delta(y) = y \otimes 1 + 1 \otimes y$$

and counit

$$\varepsilon(x) = 1$$

$$\varepsilon(y) = 0$$

Since $\mathcal{O}_q(k^2)$ is a free algebra, Δ and ε define algebra morphisms. What we have to check is that $(\mathcal{O}_q(k^2), \Delta, \varepsilon)$ is a coalgebra. By construction $\Delta(x^i y^j) = \mu_\otimes(\Delta(x)^i \otimes \Delta(y)^j)$ where $\mu_\otimes$ is the multiplicationin the tensor algebra. For the coassociativity,

$$(\Delta \otimes 1)\Delta(x^i y^j) = (\Delta \otimes 1)\mu_\otimes(\Delta(x)^i \otimes \Delta(y)^j) = \mu_\otimes((\Delta \otimes 1)\Delta(x^i) \otimes (\Delta \otimes 1)\Delta(y^j)).$$

By Example 1.5(iv), this is equal to

$$\mu_\otimes((1 \otimes \Delta)\Delta(x^i) \otimes (1 \otimes \Delta)\Delta(y^j)) = (1 \otimes \Delta)\mu_\otimes(\Delta(x)^i \otimes \Delta(y)^j) = (1 \otimes \Delta)\Delta(x^i y^j).$$

Hence Δ is coassociative. Now, also using Example 1.5(iv),

$$(\varepsilon \otimes 1)\Delta(x^i y^j) = (\varepsilon \otimes 1)\mu_\otimes(\Delta(x^i) \otimes \Delta(y^j)) = \mu_\otimes((\varepsilon \otimes 1)\Delta(x^i) \otimes (\varepsilon \otimes 1)\Delta(y^j))$$

$$= \mu_\otimes((1 \otimes x^i) \otimes (1 \otimes y^j)) = 1 \otimes x^i y^j.$$

Thus $(\mathcal{O}_q(k^2), \Delta, \varepsilon)$ is a coalgebra and so $\mathcal{O}_q(k^2)$ is a bialgebra.

Definition 1.16. Let B be a bialgebra. A subspace I of B is a biideal if I is both, an ideal and a coideal.

Example 1.17. Consider the bialgebra $\mathcal{O}(\mathrm{Mat}_n(k))$ (Example 1.15(ii)). Let $D \in \mathcal{O}(\mathrm{Mat}_n(k))$ be the determinant function. Then $\varepsilon(D) = 1$ and $\Delta(D) = D \otimes D$. For, consider the isomorphism

$$\mathcal{O}(\mathrm{Mat}_n(k)) \otimes \mathcal{O}(\mathrm{Mat}_n(k)) \xrightarrow[\cong]{\varphi} \mathcal{O}(\mathrm{Mat}_n(k) \times \mathrm{Mat}_n(k)) .$$

Since D commutes with products, $\Delta(D) = \varphi^{-1}(Dm)$ and $D \otimes D$ have the same image under the isomorphism φ. Therefore, $\Delta(D) = D \otimes D$. Let $\langle D - 1\rangle$ be the ideal generated by $D - 1$. We have that

$$\Delta(D-1) = (D-1) \otimes (D-1) \subseteq \langle D-1\rangle \otimes \mathcal{O}(\mathrm{Mat}_n(k)) + \mathcal{O}(\mathrm{Mat}_n(k)) \otimes \langle D-1\rangle .$$

And $\varepsilon(D-1) = D(I_n) - 1 = 1 - 1 = 0$. Thus $\langle D - 1\rangle$ is a biideal of $\mathcal{O}(\mathrm{Mat}_n(k))$.

Definition 1.18. Let B and B' be two bialgebras. A k-morphism $f : B \longrightarrow B'$ is a morphism of bialgebras if f is both, an algebra morphism and a coalgebra morphism.

Proposition 1.19. *Let $(B, \mu, u, \Delta, \varepsilon)$ be a bialgebra. Then ε induces a left (right) B-module structure on k. This module will be denoted by ${}_{\varepsilon}k$ (k_{ε}).*

Proof. Let $b \in B$ and $\alpha \in k$. Define $\alpha.b = \varepsilon(b)\alpha$. Since ε is an algebra morphism, k is B-module. □

The comultiplication in a bialgebra $(B, \mu, u, \Delta, \varepsilon)$ allows tensor products of B-modules to be made into B-modules. Suppose V and W are left B-modules and view the module multiplication as algebra homomorphism $m_V : B \longrightarrow \mathrm{End}_k(V)$ and $m_W : B \longrightarrow \mathrm{End}_k(W)$. Then there is an algebra homomorphism

$$B \xrightarrow{\Delta} B \otimes B \xrightarrow{m_V \otimes m_W} \mathrm{End}_k(V) \otimes \mathrm{End}_k(W) \hookrightarrow \mathrm{End}_k(V \otimes W),$$

which turns $V \otimes W$ into a left B-module. The formula for the module multiplication is $b(v \otimes w) = \sum b_1 v \otimes b_2 w$ where $\Delta(b) = \sum b_1 \otimes b_2$.

Definition 1.20. A bialgebra $(H, \mu, u, \Delta, \varepsilon)$ is a Hopf algebra if there exists a linear map $S : H \longrightarrow H$ such that

$$S * I_H = u\varepsilon = I_H * S,$$

that is, S is the inverse of the identity I_H in the convolution product. S is called the antipode of H.

By the definition we can see that

$$(S * I_H)(h) = \sum S(h_1)h_2 = \varepsilon(h)1_H \tag{1.5}$$

and

$$(I_H * S)(h) = \sum h_1 S(h_2) = \varepsilon(h)1_H \tag{1.6}$$

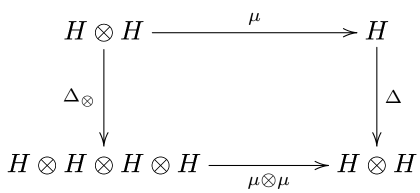
Definition 1.21. Let H and G be two Hopf algebras. A k-morphism $f : H \longrightarrow G$ is a morphism of Hopf algebras if f is both, an algebra and a coalgebra morphism such that $fS_H = S_G f$.

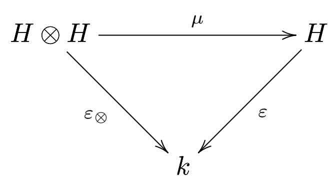
Definition 1.22. An ideal I of H is a Hopf ideal if I is a coideal and $S(I) \subseteq I$.

Let I be a Hopf ideal of a Hopf algebra H. Since I is a coideal then H/I is a bialgebra. Moreover, since $S(I) \subseteq I$, pass to S/I. Hence S/I is a Hopf algebra.

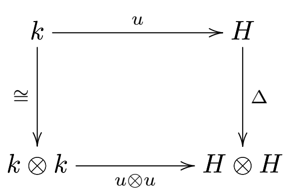
Remark 1.23. If $f : H \longrightarrow G$ is a morphism of Hopf algebras then $\operatorname{Ker} f$ is a Hopf ideal. For, since f is a coalgebra morphism $\operatorname{Ker} f$ is a coideal. Now, $f(S_H)(\operatorname{Ker} f) = S_G(f(\operatorname{Ker} f)) = 0$. Thus, $S_H(\operatorname{Ker} f) \subseteq \operatorname{Ker} f$. Taking the canonical projection, every Hopf ideal is a kernel.

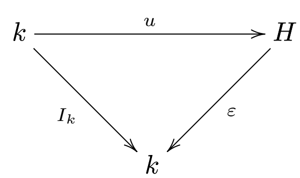
Example 1.24. (i) Let $H = k[x, x^{-1}]$. We know that H is an algebra. By Example 1.5.(ii) H is a coalgebra with comultiplication $\Delta(x^i) = x^i \otimes x^i$ and counit $\varepsilon(x^i) = 1$ for all $i \in \mathbb{Z}$. Let us see that μ and u are morphism of coalgebras. That is, we have to see that the following diagrams commute





and





So,

$$\begin{aligned}(\mu \otimes \mu)\Delta_{\otimes}(x^i \otimes x^j) &= (\mu \otimes \mu)(x^i \otimes x^j \otimes x^i \otimes x^j)\\ &= x^i x^j \otimes x^i x^j\\ &= \Delta(x^i x^j)\\ &= \Delta\mu(x^i \otimes x^j)\end{aligned}$$

and

$$\begin{aligned}\varepsilon\mu(x^{\otimes} x^j) &= \varepsilon(x^i x^j)\\ &= 1\\ &= \varepsilon_{\otimes}(x^i \otimes x^j)\end{aligned}$$

Thus, μ is an coalgebra morphism.

Now, for u,

$$\begin{aligned}\Delta u(1) &= \Delta(1_H)\\ &= 1_H \otimes 1_H\\ &= (u\otimes u)(1)\end{aligned}$$

and

$$\begin{aligned}\varepsilon u(1) &= \varepsilon(1_H)\\ &= 1\end{aligned}$$

Thus, u is a coalgebra morphism. Hence, H is a bialgebra. Define $S : H \longrightarrow H$ as $S(x^i) = x^{-1}$. Then

$$(S * I_H)(x^i) = S(x^i)x^i = x^{-1}x^i = 1_H = u\varepsilon(x^i)$$
$$(I_H * S)(x^i) = x^i S(x^i) = x^i x^{-1} = 1_H = u\varepsilon(x^i)$$

Thus H is a Hopf algebra.

(ii) The bialgebra $k[G]$ with comultiplication $\Delta(g) = g\otimes g$ and counit $\varepsilon(g) = 1$, is a Hopf algebra with antipode $S : k[G] \longrightarrow k[G]$ defined by $S(g) = g^{-1}$. For instance, the simplest infinite noncommutative example is the group algebra over k of the infinite dihedral group

$$G = \left\langle a, x \mid xax = a^{-1},\ x^2 = 1\right\rangle.$$

In this case $k[G]$ is the k-algebra generated by a, a^{-1}, x subject to the above relations and $aa^{-1} = 1 = a^{-1}a$.

(iii) Let (G, e) be a group. Then $\mathcal{F}(G) = \{f : G \longrightarrow k\}$ is an algebra with the point-wise multiplication. Moreover $\mathcal{F}(G)$ is a Hopf algebra, the counit and the antipode are defined as

$$\varepsilon(f) = f(e)$$
$$S(f)(g) = f(g^{-1})$$

Note that $\mathcal{F}(G\times G) \cong \mathcal{F}(G)\otimes\mathcal{F}(G)$ as k-algebras, this isomorphism sends $f_1 \otimes f_2$ to the function defined by $(g, h) \longmapsto f_1(g)f_2(h)$. The comultiplication is given by

$$\Delta : \mathcal{F}(G) \xrightarrow{\ \_\circ\mu\ } \mathcal{F}(G\times G) \xrightarrow{\ \cong\ } \mathcal{F}(G)\otimes\mathcal{F}(G)\ .$$

(iv) By Example 1.15(ii), $\mathcal{O}(\mathrm{Mat}_n(k))$ is a bialgebra. Consider the factor bialgebra given by the biideal $\langle D-1\rangle$ (Example 1.17).

$$\mathcal{O}(Sl_n(k)) = \mathcal{O}(\mathrm{Mat}_n(k))/\left\langle D-1\right\rangle.$$

Recall that if A is a matrix with nonzero determinant, the inverse of A can be computed as $A^{-1} = \frac{1}{D(A)}\mathrm{adj}(A)$, where $\mathrm{adj}(A)$ is the matrix of cofactors.

Define $S : \mathcal{O}(Sl_n(k)) \longrightarrow \mathcal{O}(Sl_n(k))$ as SX_{ij} the ij-entry of $(X_{ij})^{-1}$ (modulo $D-1$). Then,

$$(S * I_{\mathcal{O}(Sl_2(k))})(X_{ij}) = \sum_{k=1}^{n} S(X_{ik})X_{kj} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases} = \varepsilon(X_{ij})1.$$

Thus, S is an antipode and so $\mathcal{O}(Sl_n(k))$ is a Hopf algebra. In particular $\mathcal{O}(Sl_2(k)) = K[a, b, c, d \mid ad-bc = 1]$ where we have written a for X_{11}, b for X_{12} and so on. Hence,

	a	b	c	d
Δ	$a\otimes a+b\otimes c$	$a\otimes b+b\otimes d$	$c\otimes a+d\otimes c$	$c\otimes b+d\otimes d$
ε	1	0	0	1
S	d	$-b$	$-c$	$a.$

(v) Consider the ring of fractions of $\mathcal{O}(\mathrm{Mat}_n(k))$:

$$\mathcal{O}(GL_n(k)) = \mathcal{O}(\mathrm{Mat}_n(k))[D^{-1}].$$

So, a canonical element in $\mathcal{O}(GL_n(k))$ has the form $\frac{f}{D^n}$ with $f \in \mathcal{O}(\mathrm{Mat}_n(k))$ and $n > 0$. Let $\varphi : \mathcal{O}(\mathrm{Mat}_n(k)) \longrightarrow \mathcal{O}(GL_n(k))$ be the localization morphism. Note that $(\varphi\otimes\varphi)\Delta(D) = \frac{1}{D}\otimes\frac{1}{D}$. Hence $(\varphi\otimes\varphi)\Delta(D)$ is invertible in $\mathcal{O}(GL_n(k)) \otimes \mathcal{O}(GL_n(k))$. Thus, there exists a unique algebra morphism $\Delta_D : \mathcal{O}(GL_n(k)) \longrightarrow \mathcal{O}(GL_n(k)) \otimes \mathcal{O}(GL_n(k))$. We can see that $\Delta_D(\frac{f}{D^n}) = \sum \frac{f_1}{D^n}\otimes\frac{f_2}{D^n}$ where $\Delta(f) = \sum f_1 \otimes f_2$. On the other hand. since $\varepsilon(D) = 1$, there exists a unique algebra morphism $\varepsilon_D : \mathcal{O}(GL_n(k)) \longrightarrow k$ such that $\varepsilon(\frac{f}{D^n}) = f(I_n)$. It is not difficult to see that $(\mathcal{O}(GL_n(k)), \Delta_D, \varepsilon_D)$ is a bialgebra. Moreover, $\mathcal{O}(GL_n(k))$ is a Hopf algebra where the antipode can be defined as in the previous example.

(vi) Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n\geq 0}\mathfrak{g}^{\otimes n}$ by the two sided ideal generated by the elements $x\otimes y - y\otimes x - [x,y]$ for all $x, y \in \mathfrak{g}$. The Poincaré-Birkhoff-Witt Theorem [5] asserts that if $\{x_i\}_I$ is any basis for $\mathfrak{g}$, where the index set I is totally ordered, the set of monomials $\{x_{i_1}x_{i_2}\cdots x_{i_k}\}$ where $k \geq 1$ and $i_1 \leq i_2 \leq \cdots \leq i_k$, is a basis for $\mathcal{U}(\mathfrak{g})$. The composition of the natural maps $\mathfrak{g}\longrightarrow T(\mathfrak{g})\longrightarrow\mathcal{U}(\mathfrak{g})$ is an embedding. Hence we can identify $\mathfrak{g}$ with its image in $\mathcal{U}(\mathfrak{g})$. To define an algebra morphism from $\mathcal{U}(\mathfrak{g})$, it is enough to define it in $\mathfrak{g}$, since $\mathfrak{g}$ generates $T(\mathfrak{g})$ as algebra and check that the morphism pass to the factor algebra. We define the following morphisms:

$$\Delta(x) = x\otimes 1 + 1\otimes x$$

$$\varepsilon(x) = 0,$$

for $x \in \mathfrak{g}$. Let us see that these algebra morphisms pass to $\mathcal{U}(\mathfrak{g})$.

$$\begin{aligned}\Delta(xy-yx) &= \Delta(x)\Delta(y) - \Delta(y)\Delta(x)\\ &= (x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y) - (y\otimes 1+1\otimes y)(x\otimes 1+1\otimes x)\\ &= (xy\otimes 1 + x\otimes y + y\otimes x + 1\otimes xy) - (yx\otimes 1 + y\otimes x + x\otimes y + 1\otimes yx)\\ &= (xy-yx)\otimes 1 + 1\otimes(xy-yx)\\ &= [x,y]\otimes 1 + 1\otimes [x,y]\\ &= \Delta([x,y])\end{aligned}$$

Hence Δ defines an algebra morphism from $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$. It is clear that $\varepsilon : \mathcal{U}(\mathfrak{g}) \longrightarrow k$ is an algebra homomorphism. Now, define $S : T(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})^{op}$ by the rule

$$S(x) = -x.$$

Then, S is an anti-homomorphism $S : T(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$. Hence,

$$S(xy-yx) = S(xy)-S(yx) = S(y)S(x)-S(x)S(y) = yx-xy = -[x,y] = S([x,y]).$$

Hence, $S:\mathcal{U}(\mathfrak{g})\longrightarrow\mathcal{U}(\mathfrak{g})$. Moreover, for any $x\in\mathfrak{g}$

$$(S*I_{\mathcal{U}(\mathfrak{g})})(x)=S(x)1+S(1)x=-x+x=0=u\varepsilon(x)$$

Analogously,

$$(I_{\mathcal{U}(\mathfrak{g})}*S)(x)=xS(1)+1S(x)=x-x=0=u\varepsilon(x)$$

Let $y\in\mathfrak{g}$ and $x\in\mathcal{U}(\mathfrak{g})$, then:

$$\begin{aligned}(S*I_{\mathcal{U}(\mathfrak{g})})(xy)&=\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})\Delta(xy)\\&=\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})(\Delta(x)\Delta(y))\\&=\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})(\Delta(x)(y\otimes 1+1\otimes y))\\&=\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})(\Delta(x)(y\otimes 1)+\Delta(x)(1\otimes y))\\&=\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})(\Delta(x)(y\otimes 1))+\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})(\Delta(x)(1\otimes y))\\&=\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})((\sum x_1\otimes x_2)(y\otimes 1))+\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})((\sum x_1\otimes x_2)(1\otimes y))\\&=\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})(\sum x_1y\otimes x_2)+\mu(S\otimes I_{\mathcal{U}(\mathfrak{g})})(\sum x_1\otimes x_2y)\\&=\sum S(x_1y)x_2+\sum S(x_1)x_2y\\&=\sum S(y)S(x_1)x_2+\sum S(x_1)x_2y\\&=S(y)(S*I_{\mathcal{U}(\mathfrak{g})})(x)+(S*I_{\mathcal{U}(\mathfrak{g})})(x)y\\&=-y(S*I_{\mathcal{U}(\mathfrak{g})})(x)+(S*I_{\mathcal{U}(\mathfrak{g})})(x)y\end{aligned}$$

In particular, if $x,y\in\mathfrak{g}$, then $(S*I_{\mathcal{U}(\mathfrak{g})})(xy)=0$. Therefore, by induction $(S*I_{\mathcal{U}(\mathfrak{g})})(x_{i_1}x_{i_2}\cdots x_{i_k})=0$ for every monoid. Analogously $(I_{\mathcal{U}(\mathfrak{g})}*S)(x_{i_1}x_{i_2}\cdots x_{i_k})=0$. Thus S is an antipode, and so $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra.

Consider $\mathfrak{g}=\mathfrak{sl}_2(k)$, with k-basis,

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Then $\mathcal{U}(\mathfrak{g})$ is the k-algebra with generators e,f and h and relations

$$he-eh=2e\quad hf-fh=-2f\quad ef-fe=h$$

Proposition 1.25. *Let $(H,\mu,u,\Delta,\varepsilon,S)$ be a Hopf algebra.*

(1) *$S(gh)=S(h)S(g)$ and $S(1_H)=1_H$.*
(2) *If H is either commutative or co-commutative then $S^2=I_H$.*

Proof. 1. We have that $H\otimes H$ is a coalgebra, then $\mathrm{Hom}_k(H\otimes H,H)$ is an algebra. Let $m,\rho\in\mathrm{Hom}_k(H\otimes H,H)$ as follows:

$$m(g\otimes h)=S(h)S(g)$$

$$\rho(g\otimes h)=S(gh)$$

We claim that $\rho * \mu = \mu * m = u\varepsilon_{\otimes}$.

$$\begin{aligned} \rho * \mu(g \otimes h) &= \mu(\rho \otimes \mu)\Delta_{\otimes}(g \otimes h) \\ &= \mu(S \otimes I_H)(\mu \otimes \mu)\Delta_{\otimes}(g \otimes h) \\ \Delta \text{ is an alg. morph. } &= \mu(S \otimes I_H)\Delta\mu(g \otimes h) \\ &= \mu(S \otimes I_H)\Delta(gh) \\ &= \mu(S \otimes I_H)(\sum (gh)_1 \otimes (gh)_2) \\ &= \sum S((gh)_1)(gh)_2 \\ \text{By (1.5)} &= \varepsilon(gh)1_H \\ &= u\varepsilon_{\otimes}(g \otimes h) \end{aligned}$$

On the other hand,

$$\begin{aligned} \mu * m(g \otimes h) &= \mu(\mu \otimes m)\Delta_{\otimes}(g \otimes h) \\ &= \mu(\mu \otimes \mu)(I_{H \otimes H} \otimes (S \otimes S))(I_{H \otimes H} \otimes \tau)(\sum g_1 \otimes h_1 \otimes g_2 \otimes h_2) \\ &= \mu(\mu \otimes \mu)(I_{H \otimes H} \otimes (S \otimes S))(\sum g_1 \otimes h_1 \otimes h_2 \otimes g_2) \\ &= \mu(\mu \otimes \mu)(\sum g_1 \otimes h_1 \otimes S(h_2) \otimes S(g_2)) \\ &= \sum g_1 h_1 S(h_2) S(g_2) \\ \text{By (1.6) } &= \sum g_1 \varepsilon(h) S(g_2) \\ \text{By (1.6) } &= \varepsilon(g)\varepsilon(h)1_H \\ &= u\varepsilon_{\otimes}(g \otimes h) \end{aligned}$$

Hence ρ and m are inverses of μ in the convolution product. Thus, $\rho = m$. Now, we have that $S * I_H = u\varepsilon$, hence

$$1_H = \varepsilon(1_H)1_H = u\varepsilon(1_H) = (S * I_H)(1_H) = S(1_H)1_H = S(1_H)$$

2. Suppose H is commutative. Then,

$$\begin{aligned} (S * S^2)(g) &= \mu(S \otimes S^2)\Delta(g) \\ &= \mu(S \otimes S^2)(\sum g_1 \otimes g_2) \\ &= \sum S(g_1)S^2(g_2) \\ &= S(\sum S(g_2)g_1) \\ H \text{ is commutative } &= S(\sum g_1 S(g_2)) \\ &= S(\varepsilon(g)) \\ &= \varepsilon(g)S(1_H) \\ &= \varepsilon(g)1_H \\ &= u\varepsilon(g) \end{aligned}$$

Analogously, $S^2 * S = u\varepsilon$. Hence S^2 is an inverse of S in the convolution product. Thus $S^2 = I_H$. □

References: [1], [2], [3], [4].

2. Margin notes and Exercises

Pag. 5, Eq (3).

In the ring $\mathrm{Mat}_n(k)$ we have additional structure given by the multiplication of row or column vectors by matrices. These give morphisms

$$R : k^n \times \mathrm{Mat}_n(k) \longrightarrow k^n \text{ and } C : \mathrm{Mat}_n(k) \times k^n \longrightarrow k^n.$$

Hence R and C induces algebra homomorphisms in the coordinate rings:

$$\mathcal{O}(k^n) \longrightarrow \mathcal{O}(k^n \times \mathrm{Mat}_n(k)) \cong \mathcal{O}(k^n) \otimes \mathcal{O}(\mathrm{Mat}_n(k))$$

and

$$\mathcal{O}(k^n) \longrightarrow \mathcal{O}(\mathrm{Mat}_n(k) \times k^n) \cong \mathcal{O}(\mathrm{Mat}_n(k)) \otimes \mathcal{O}(k^n)$$

given by precompose R and C respectively. Let us check what the first morphism is doing. Consider the coordinate function $x_j \in \mathcal{O}(k^n)$ and let $(a_1, ..., a_n) \in k^n$ and $(b_{ij}) \in \mathrm{Mat}_n(k)$. Then

$$x_j R((a_1, ..., a_n), (b_{ij})) = x_j(\sum_{i=1}^{n} a_i b_{i1}, ..., \sum_{i=1}^{n} a_i b_{in}) = \sum_{i=1}^{n} a_i b_{ij}.$$

This implies that

$$x_j \longmapsto \sum_{i=1}^{n} x_i \otimes X_{ij} \in \mathcal{O}(k^n) \otimes \mathcal{O}(\mathrm{Mat}_n(k)).$$

Analogously,

$$x_i \longmapsto \sum_{j=1}^{n} X_{ij} \otimes x_j \in \mathcal{O}(\mathrm{Mat}_n(k)) \otimes \mathcal{O}(k^n).$$

which are equations (3) in [1, Pp. 5].

Example I.1.6 and Exercise I.1.C

We want to quantize $Sl_n(k)$ (Example 1.24(iv)) but first we have to quantize $\mathrm{Mat}_n(k)$ (Example 1.15(ii)). For, we need a bialgebra $(B, \mu, u, \Delta, \varepsilon)$, generated as k-algebra by elements X_{ij} satisfying:

$$\Delta(X_{ij}) = \sum_{\ell=1}^{n} X_{i\ell} \otimes X_{\ell j}$$

and

$$\varepsilon(X_{ij}) = \delta_{ij};$$

which supports k-algebra homomorphisms

$$R : \mathcal{O}_q(k^n) \longrightarrow \mathcal{O}_q(k^n) \otimes B \ \text{ and } \ C : \mathcal{O}_q(k^n) \longrightarrow B \otimes \mathcal{O}_q(k^n)$$

satisfying:

$$x_j \longmapsto \sum_{i=1}^{n} x_i \otimes X_{ij} \ \text{ and } \ x_i \longmapsto \sum_{j=1}^{n} X_{ij} \otimes x_j.$$

where $\mathcal{O}_q(k^n) = k \langle x_1, ..., x_n \mid x_i x_j = q x_j x_i \text{ for } i < j \rangle$ the quantum affine n-space.

For convenience, let us check the case $n = 2$. Consider $\rho_i : \mathcal{O}_q(k^n) \longrightarrow k$ defined as $\rho_i(x_j) = \delta_{ij}$ for $i = 1, 2$. It is clear that ρ_i is a k-algebra homomorphism for all $1 \leq i \leq n$. Then we have the compositions

$$(\rho_i \otimes 1)R : \mathcal{O}_q(k^2) \longrightarrow B$$

and

$$(1 \otimes \rho_j)C : \mathcal{O}_q(k^2) \longrightarrow B.$$

Note that $(\rho_i \otimes 1)R(x_j) = X_{ij}$ and $(1 \otimes \rho_j)C(x_i) = X_{ij}$. Hence,

$$\begin{aligned} X_{1\ell}X_{2\ell} &= (1 \otimes \rho_\ell)C(x_1x_2) = (1 \otimes \rho_\ell)C(qx_2x_1) = qX_{2\ell}X_{1\ell} \\ X_{\ell 1}X_{\ell 2} &= (\rho_\ell \otimes 1)R(x_1x_2) = (\rho_\ell \otimes 1)R(qx_2x_1) = qX_{\ell 2}X_{\ell 1} \end{aligned} \tag{2.1}$$

for $\ell = 1, 2$.

On the other hand,

$$\begin{aligned} C(x_1x_2) &= (X_{11} \otimes x_1 + X_{12} \otimes x_2)(X_{21} \otimes x_1 + X_{22} \otimes x_2) \\ &= X_{11}X_{21} \otimes x_1^2 + X_{12}X_{21} \otimes x_2x_1 + X_{11}X_{22} \otimes x_1x_2 + X_{12}X_{22} \otimes x_2^2 \\ &= X_{11}X_{21} \otimes x_1^2 + (q^{-1}X_{12}X_{21} + X_{11}X_{22}) \otimes x_1x_2 + X_{12}X_{22} \otimes x_2^2 \\ qC(x_2x_1) &= q(X_{21} \otimes x_1 + X_{22} \otimes x_2)(X_{11} \otimes x_1 + X_{12} \otimes x_2) \\ &= qX_{21}X_{11} \otimes x_1^2 + qX_{22}X_{11} \otimes x_2x_1 + qX_{21}X_{12} \otimes x_1x_2 + qX_{22}X_{12} \otimes x_2^2 \\ &= X_{11}X_{21} \otimes x_1^2 + (X_{22}X_{11} + qX_{21}X_{12}) \otimes x_1x_2 + X_{12}X_{22} \otimes x_2^2 \end{aligned}$$

Then $(q^{-1}X_{12}X_{21} + X_{11}X_{22}) - (X_{22}X_{11} + qX_{21}X_{12}) \otimes x_1x_2 = 0$. Analogously, using R we have $x_1x_2 \otimes (q^{-1}X_{21}X_{12} + X_{11}X_{22}) - (X_{22}X_{11} + qX_{12}X_{21}) = 0$.

Remark 2.1. Let V and W be k-vector spaces. Take $0 \neq w \in W$ then the k-morphism $(w \otimes \_) : V \longrightarrow W \otimes V$ is injective.

By the Remark we have that

$$X_{11}X_{22} - X_{22}X_{11} = qX_{21}X_{12} - q^{-1}X_{12}X_{21} = qX_{12}X_{21} - q^{-1}X_{21}X_{12}. \tag{2.2}$$

Now, suppose $q^2 \neq -1$. Then multiplying equation 2.2 by q we get

$$q^2X_{21}X_{12} - X_{12}X_{21} = q^2X_{12}X_{21} - X_{21}X_{12}$$

and so

$$-q^2(X_{12}X_{21} - X_{21}X_{12}) = X_{12}X_{21} - X_{21}X_{12}.$$

It implies that

$$X_{12}X_{21} = X_{21}X_{12} \tag{2.3}$$

Therefore

$$X_{11}X_{22} - X_{22}X_{11} = qX_{21}X_{12} - q^{-1}X_{12}X_{21} = (q - q^{-1})X_{12}X_{21} \tag{2.4}$$

Exercise I.1.D

Let B the k-algebra given by generators $X_{11}, X_{12}, X_{21}, X_{22}$ and the relations 2.1,2.2,2.3 and 2.4. Let us see that the k-algebra morphism

$$\Delta : k\langle X_{11}, X_{12}, X_{21}, X_{22}\rangle \longrightarrow B \otimes B$$

defined in generators as $\Delta(X_{ij}) = X_{i1}\otimes X_{1j} + X_{i2}\otimes X_{2j}$ respects the relations in B.

$$\begin{aligned}
\Delta(X_{\ell 1}X_{\ell 2}) &= (X_{\ell 1}\otimes X_{11} + X_{\ell 2}\otimes X_{21})(X_{\ell 1}\otimes X_{12} + X_{\ell 2}\otimes X_{22})\\
&= X_{\ell 1}^2\otimes X_{11}X_{12} + X_{\ell 2}X_{\ell 1}\otimes X_{21}X_{12} + X_{\ell 1}X_{\ell 2}\otimes X_{11}X_{22} + X_{\ell 2}^2\otimes X_{21}X_{22}\\
\text{Using 2.1} &= X_{\ell 1}^2\otimes X_{12}X_{11} + X_{\ell 2}X_{\ell 1}\otimes X_{21}X_{12} + qX_{\ell 2}X_{\ell 1}\otimes X_{11}X_{22} + X_{\ell 2}^2\otimes X_{22}X_{21}\\
&= X_{\ell 1}^2\otimes X_{12}X_{11} + X_{\ell 2}X_{\ell 1}\otimes (X_{21}X_{12} + qX_{11}X_{22}) + X_{\ell 2}^2\otimes X_{22}X_{21}
\end{aligned}$$

On the other hand

$$\begin{aligned}
q\Delta(X_{\ell 2}X_{\ell 1}) &= q\left[(X_{\ell 1}\otimes X_{12} + X_{\ell 2}\otimes X_{22})(X_{\ell 1}\otimes X_{11} + X_{\ell 2}\otimes X_{21})\right]\\
&= q\left[X_{\ell 1}^2\otimes X_{12}X_{11} + X_{\ell 2}X_{\ell 1}\otimes X_{22}X_{11} + X_{\ell 1}X_{\ell 2}\otimes X_{12}X_{21} + X_{\ell 2}^2\otimes X_{22}X_{21}\right]\\
\text{Using 2.1} &= q\left[X_{\ell 1}^2\otimes X_{12}X_{11} + X_{\ell 2}X_{\ell 1}\otimes X_{22}X_{11} + qX_{\ell 2}X_{\ell 1}\otimes X_{12}X_{21} + X_{\ell 2}^2\otimes X_{22}X_{21}\right]\\
&= q\left[X_{\ell 1}^2\otimes X_{12}X_{11} + X_{\ell 2}X_{\ell 1}\otimes (X_{22}X_{11} + qX_{12}X_{21}) + X_{\ell 2}^2\otimes X_{22}X_{21}\right]\\
\text{Using 2.2} &= q\left[X_{\ell 1}^2\otimes X_{12}X_{11} + X_{\ell 2}X_{\ell 1}\otimes (X_{11}X_{22} + q^{-1}X_{21}X_{12}) + X_{\ell 2}^2\otimes X_{22}X_{21}\right]
\end{aligned}$$

Thus $\Delta(X_{\ell 1}X_{\ell 2}) = q\Delta(X_{\ell 2}X_{\ell 1})$ for $\ell = 1, 2$.

$$\begin{aligned}
\Delta(X_{11}X_{22}) &= (X_{11}\otimes X_{11} + X_{12}\otimes X_{21})(X_{21}\otimes X_{12} + X_{22}\otimes X_{22})\\
&= X_{11}X_{21}\otimes X_{11}X_{12} + X_{12}X_{21}\otimes X_{21}X_{12} + X_{11}X_{22}\otimes X_{11}X_{22} + X_{12}X_{22}\otimes X_{21}X_{22}\\
\text{Using 2.1} &= q^2X_{21}X_{11}\otimes X_{12}X_{11} + X_{12}X_{21}\otimes X_{21}X_{12} + X_{11}X_{22}\otimes X_{11}X_{22} + q^2X_{22}X_{12}\otimes X_{22}X_{21}
\end{aligned}$$

$$\begin{aligned}
\Delta(X_{22}X_{11}) &= (X_{21}\otimes X_{12} + X_{22}\otimes X_{22})(X_{11}\otimes X_{11} + X_{12}\otimes X_{21})\\
&= X_{21}X_{11}\otimes X_{12}X_{11} + X_{21}X_{12}\otimes X_{12}X_{21} + X_{22}X_{11}\otimes X_{22}X_{11} + X_{22}X_{12}\otimes X_{22}X_{21}
\end{aligned}$$

$$\begin{aligned}
q\Delta(X_{21}X_{12}) &= q(X_{21}\otimes X_{11} + X_{22}\otimes X_{21})(X_{11}\otimes X_{12} + X_{12}\otimes X_{22})\\
&= q\left[X_{21}X_{11}\otimes X_{11}X_{12} + X_{22}X_{11}\otimes X_{21}X_{12} + X_{21}X_{12}\otimes X_{11}X_{22} + X_{22}X_{12}\otimes X_{21}X_{22}\right]
\end{aligned}$$

$$\begin{aligned}
q^{-1}\Delta(X_{12}X_{21}) &= q^{-1}(X_{11}\otimes X_{12} + X_{12}\otimes X_{22})(X_{21}\otimes X_{11} + X_{22}\otimes X_{21})\\
&= q^{-1}\left[X_{11}X_{21}\otimes X_{12}X_{11} + X_{11}X_{22}\otimes X_{12}X_{21} + X_{12}X_{21}\otimes X_{22}X_{11} + X_{12}X_{22}\otimes X_{22}X_{21}\right]\\
\text{Using 2.1} &= q^{-1}\left[X_{21}X_{11}\otimes X_{11}X_{12} + X_{11}X_{22}\otimes X_{12}X_{21} + X_{12}X_{21}\otimes X_{22}X_{11} + X_{22}X_{12}\otimes X_{21}X_{22}\right]
\end{aligned}$$

In this way, the other relations can be checked. The k-algebra morphism

$$\varepsilon : k\langle X_{11}, X_{12}, X_{21}, X_{22}\rangle \longrightarrow k$$

defined in generators as $\Delta(X_{ij}) = \delta_{ij}$ respects the relations in B. If fact, ε sends all the relations in B to zero. Hence, we have a bialgebra denoted $\mathcal{O}_q(\mathrm{Mat}_2(k))$ called *the quantum* 2×2 *matrix algebra*. Now let us see that the maps

$$R : \mathcal{O}_q(k^2) \longrightarrow \mathcal{O}_q(k^2)\otimes B \quad\text{and}\quad C : \mathcal{O}_q(k^2) \longrightarrow B\otimes \mathcal{O}_q(k^2)$$

given by:

$$x_j \longmapsto x_1\otimes X_{1j} + x_2\otimes X_{2j} \quad\text{and}\quad x_i \longmapsto X_{i1}\otimes x_1 + X_{i2}\otimes x_2,$$

are well defined, that is, $R(x_1x_2) = R(qx_2x_1)$ and $C(x_1x_2) = C(qx_2x_1)$.

$$\begin{aligned} R(x_1x_2) &= (x_1 \otimes X_{11} + x_2 \otimes X_{21})(x_1 \otimes X_{12} + x_2 \otimes X_{22}) \\ &= x_1^2 \otimes X_{11}X_{12} + x_2x_1 \otimes X_{21}X_{12} + x_1x_2 \otimes X_{11}X_{22} + x_2^2 \otimes X_{21}X_{22} \\ \text{Using 2.1} &= qx_1^2 \otimes X_{12}X_{11} + x_2x_1 \otimes X_{21}X_{12} + qx_2x_1 \otimes X_{11}X_{22} + qx_2^2 \otimes X_{22}X_{21} \\ &= qx_1^2 \otimes X_{12}X_{11} + x_2x_1 \otimes (X_{21}X_{12} + qX_{11}X_{22}) + qx_2^2 \otimes X_{22}X_{21} \\ \text{Using 2.2} &= qx_1^2 \otimes X_{12}X_{11} + x_2x_1 \otimes (q^2X_{12}X_{21} + qX_{22}X_{11}) + qx_2^2 \otimes X_{22}X_{21} \end{aligned}$$

On the other hand

$$\begin{aligned} qR(x_2x_1) &= (x_1 \otimes X_{12} + x_2 \otimes X_{22})(x_1 \otimes X_{11} + x_2 \otimes X_{21}) \\ &= q\left[x_1^2 \otimes X_{12}X_{11} + x_2x_1 \otimes X_{22}X_{11} + x_1x_2 \otimes X_{12}X_{21} + x_2^2 \otimes X_{22}X_{21}\right] \\ &= q\left[x_1^2 \otimes X_{12}X_{11} + x_2x_1 \otimes X_{22}X_{11} + qx_2x_1 \otimes X_{12}X_{21} + x_2^2 \otimes X_{22}X_{21}\right] \\ &= q\left[x_1^2 \otimes X_{12}X_{11} + x_2x_1 \otimes (X_{22}X_{11} + qX_{12}X_{21}) + x_2^2 \otimes X_{22}X_{21}\right] \end{aligned}$$

Thus $R(x_1x_2) = R(qx_2x_1)$. Analogously, $C(x_1x_2) = C(qx_2x_1)$.

Exercise I.1.E

It can be seen that in last exercise, all computations were made using equations 2.1 and 2.2. Hence, we have a bialgebra B satisfying equations 2.1 and 2.2. Now, consider the ordered monomials $X_{11}^{\bullet}X_{12}^{\bullet}X_{21}^{\bullet}X_{22}^{\bullet}$ in B. From equation 2.2 we get that $X_{21}X_{12} = q^2X_{12}X_{21} + qX_{11}X_{22} + qX_{22}X_{11}$. This implies that the monomials $X_{11}^{\bullet}X_{12}^{\bullet}X_{21}^{\bullet}X_{22}^{\bullet}$ generate $X_{21}X_{12}$ if and only if they generate $X_{22}X_{11}$.

Pag. 6, Example I.1.8

The exterior algebra $\Lambda(V)$ of a vector space V over a field k is defined as the quotient algebra of the tensor algebra $\mathcal{T}(V)$ by the two-sided ideal $\langle x \otimes x \rangle$ with $x \in V$, that is

$$\Lambda(V) = \mathcal{T}(V)/\langle x \otimes x \rangle .$$

The coset of an element $x_1 \otimes x_2 \otimes \cdots \otimes x_n$ is denoted as $x_1 \wedge x_2 \wedge \cdots \wedge x_n$. Note that if the dimension of V is n, any coset $x_1 \wedge x_2 \wedge \cdots \wedge x_\ell$ with $\ell > n$ is zero. Let us consider $\Lambda(k^2)$.

If $\{x_1, x_2\}$ is a basis of k^2, there is a k-linear map

$$k^2 \longrightarrow \mathcal{O}(\mathrm{Mat}_2(k)) \otimes \Lambda(k^2)$$

given by

$$x_1 \longmapsto X_{11} \otimes x_1 + X_{12} \otimes x_2$$

$$x_2 \longmapsto X_{21} \otimes x_1 + X_{22} \otimes x_2$$

By the universal property of $\mathcal{T}(k^2)$ there exists a unique k-algebra homomorphism

$$\mathcal{T}(k^2) \longrightarrow \mathcal{O}(\mathrm{Mat}_2(k)) \otimes \Lambda(k^2).$$

Let us see that this mapping factors through the tensor algebra.

(2.5)

$$
\begin{aligned}
x_1 \otimes x_1 \longmapsto &(X_{11} \otimes x_1 + X_{12} \otimes x_2)(X_{11} \otimes x_1 + X_{12} \otimes x_2)\\
&= X_{11}^2 \otimes x_1 \wedge x_1 + X_{11}X_{12} \otimes x_1 \wedge x_2 + X_{12}X_{11} \otimes x_2 \wedge x_1 + X_{12}^2 \otimes x_2 \wedge x_2\\
&= X_{11}X_{12} \otimes x_1 \wedge x_2 - X_{11}X_{12} \otimes x_1 \wedge x_2\\
&= 0.\\
x_1 \otimes x_2 \longmapsto &(X_{11} \otimes x_1 + X_{12} \otimes x_2)(X_{21} \otimes x_1 + X_{22} \otimes x_2)\\
&= X_{11}X_{21} \otimes x_1 \wedge x_1 + X_{11}X_{22} \otimes x_1 \wedge x_2 + X_{12}X_{21} \otimes x_2 \wedge x_1 + X_{12}X_{22} \otimes x_2 \wedge x_2\\
&= X_{11}X_{22} - X_{12}X_{21} \otimes x_1 \wedge x_2\\
x_2 \otimes x_1 \longmapsto &(X_{21} \otimes x_1 + X_{22} \otimes x_2)(X_{11} \otimes x_1 + X_{12} \otimes x_2)\\
&= X_{21}X_{11} \otimes x_1 \wedge x_1 + X_{22}X_{11} \otimes x_2 \wedge x_1 + X_{21}X_{12} \otimes x_1 \wedge x_2 + X_{22}X_{12} \otimes x_2 \wedge x_2\\
&= -(X_{11}X_{22} - X_{12}X_{21}) \otimes x_1 \wedge x_2
\end{aligned}
$$

Analogously, $x_2 \otimes x_2 \longmapsto 0$. Hence $x \otimes x \longmapsto 0$ for all $x \in k^2$. Therefore, there is a k-algebra homomorphism

$$\Lambda(k^2) \longrightarrow \mathcal{O}(\mathrm{Mat}_2(k)) \otimes \Lambda(k)$$

sending

$$x_1 \wedge x_2 \longmapsto det(X_{ij}) \otimes x_1 \wedge x_2.$$

Exercise I.1.F

The *quantum exterior algebra* $\Lambda_q(k^2)$ is defined to be the k-algebra given by generators ξ_1 and ξ_2 and relations

$$\xi_1^2 = 0 = \xi_2^2 \text{ and } \xi_2\xi_1 = -q\xi_1\xi_2. \tag{2.6}$$

Consider the map $\phi : k\langle \xi_1, \xi_2 \rangle \longrightarrow \mathcal{O}_q(\mathrm{Mat}_2(k)) \otimes \Lambda_q(k^2)$ given by

$$\phi(\xi_1) = X_{11} \otimes \xi_1 + X_{12} \otimes \xi_2 \text{ and } \phi(\xi_2) = X_{21} \otimes \xi_1 + X_{22} \otimes \xi_2.$$

Then,

$$
\begin{aligned}
\phi(\xi_1\xi_1) &= (X_11 \otimes \xi_1 + X_{12} \otimes \xi_2)(X_11 \otimes \xi_1 + X_{12} \otimes \xi_2)\\
&= X_{11}^2 \otimes \xi_1^2 + X_{12}X_{11} \otimes \xi_2\xi_1 + X_{11}X_{12} \otimes \xi_1\xi_2 + X_{12}^2 \otimes \xi_2^2\\
&= (-qX_{12}X_{11} + X_{11}X_{12}) \otimes \xi_1\xi_2\\
\text{Using } 2.1 &= 0
\end{aligned}
$$

Analogously, $\phi(\xi_2^2) = 0$.

$$
\begin{aligned}
\phi(\xi_2\xi_1) &= (X_{21} \otimes \xi_1 + X_{22} \otimes \xi_2)(X_11 \otimes \xi_1 + X_{12} \otimes \xi_2)\\
&= X_{21}X_{11} \otimes \xi_1^2 + X_{22}X_{11} \otimes \xi_2\xi_1 + X_{21}X_{12} \otimes \xi_1\xi_2 + X_{22}X_{12} \otimes \xi_2^2\\
\text{By } 2.6 &= (X_{21}X_{12} - qX_{22}X_{11}) \otimes \xi_1\xi_2\\
\text{By } 2.2 &= (q^2X_{12}X_{21} - qX_{11}X_{22}) \otimes \xi_1\xi_2.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
-q\phi(\xi_1\xi_2) &= -q(X_11 \otimes \xi_1 + X_{12} \otimes \xi_2)(X_{21} \otimes \xi_1 + X_{22} \otimes \xi_2) \\
&= -q\left[X_{11}X_{21} \otimes \xi_1^2 + X_{12}X_{21} \otimes \xi_2\xi_1 + X_{11}X_{22} \otimes \xi_1\xi_2 + X_{12}X_{22} \otimes \xi_2^2\right] \\
\text{By 2.6} &= -q(X_{11}X_{22} - qX_{12}X_{21}) \otimes \xi_1\xi_2 \\
&= (q^2X_{12}X_{21} - qX_{11}X_{22}) \otimes \xi_1\xi_2
\end{aligned}$$

Thus, we have a k-algebra homomorphism

$$\phi : \Lambda_q(k^2) \longrightarrow \mathcal{O}_q(\mathrm{Mat}_2(k)) \otimes \Lambda_q(k^2).$$

Note that

$$\phi(\xi_1\xi_2) = (X_{11}X_{22} - qX_{12}X_{21}) \otimes \xi_1\xi_2.$$

Exercise I.1.G

Now, consider the element $D_q = X_{11}X_{22} - qX_{12}X_{21} \in \mathcal{O}_q(\mathrm{Mat}_2(k))$. We claim that D_q is in the centre of $\mathcal{O}_q(\mathrm{Mat}_2(k))$.

$$\begin{aligned}
D_qX_{11} &= (X_{11}X_{22} - qX_{12}X_{21})X_{11} \\
&= X_{11}X_{22}X_{11} - qX_{12}X_{21}X_{11} \\
&= X_{11}X_{22}X_{11} - qX_{12}(q^{-1}X_{11}X_{21}) \\
&= X_{11}X_{22}X_{11} - X_{12}X_{11}X_{21} \\
&= X_{11}X_{22}X_{11} - q^{-1}X_{11}X_{12}X_{21} \\
&= X_{11}(X_{22}X_{11} - q^{-1}X_{12}X_{21}) \\
\text{By 2.2} &= X_{11}(X_{11}X_{22} - qX_{21}X_{12}) \\
\text{By 2.3} &= X_{11}(X_{11}X_{22} - qX_{12}X_{21}) \\
&= X_{11}D_q
\end{aligned}$$

$$\begin{aligned}
D_qX_{12} &= (X_{11}X_{22} - qX_{12}X_{21})X_{12} \\
&= X_{11}X_{22}X_{12} - qX_{12}X_{21}X_{12} \\
&= X_{11}(q^{-1}X_{12}X_{22}) - qX_{12}X_{21}X_{12} \\
&= X_{12}X_{11}X_{22} - qX_{12}X_{21}X_{12} \\
&= X_{12}(X_{11}X_{22} - qX_{21}X_{12}) \\
&= X_{12}(X_{11}X_{22} - qX_{12}X_{21}) \\
&= X_{12}D_q
\end{aligned}$$

In the same way, it can be seen that D_q commutes with all the generators. Thus D_q is a central element. Therefore, we can define

$$\mathcal{O}_q(Sl_2(k)) = \mathcal{O}(\mathrm{Mat}_2(k))/\langle D_q - 1\rangle$$

Exercise I.1.H

Let see that the comultiplication Δ and the counit ε in $\mathcal{O}_q(\mathrm{Mat}_2(k))$ induce a multiplication and a counit in $\mathcal{O}_q(Sl_2(k))$. For, consider $\Delta : \mathcal{O}_q(\mathrm{Mat}_2(k)) \longrightarrow \mathcal{O}_q(Sl_2(k)) \otimes \mathcal{O}_q(Sl_2(k))$ given by

$$\Delta(X_{ij}) = X_{i1} \otimes X_{1j} + X_{i2} \otimes X_{2j}.$$

Then,

$$\begin{aligned}\Delta(X_{11})\Delta(X_{22}) &= (X_{11}\otimes X_{11}+X_{12}\otimes X_{21})(X_{21}\otimes X_{12}+X_{22}\otimes X_{22})\\ &= X_{11}X_{21}\otimes X_{11}X_{12}+X_{12}X_{21}\otimes X_{21}X_{12}+X_{11}X_{22}\otimes X_{11}X_{22}+X_{12}X_{22}\otimes X_{21}X_{22}\\ q\Delta(X_{12})\Delta(X_{21}) &= q(X_{11}\otimes X_{12}+X_{12}\otimes X_{22})(X_{21}\otimes X_{11}+X_{22}\otimes X_{21})\\ &= q\left[X_{11}X_{21}\otimes X_{12}X_{11}+X_{12}X_{21}\otimes X_{22}X_{11}+X_{11}X_{22}\otimes X_{12}X_{21}+X_{12}X_{22}\otimes X_{22}X_{21}\right]\\ &= X_{11}X_{21}\otimes qX_{12}X_{11}+X_{12}X_{21}\otimes qX_{22}X_{11}+X_{11}X_{22}\otimes qX_{12}X_{21}+X_{12}X_{22}\otimes qX_{22}X_{21}\end{aligned}$$

Hence

$$\begin{aligned}\Delta(D_q) &= \Delta(X_{11}X_{22}-qX_{12}X_{21})\\ &= X_{12}X_{21}\otimes(X_{21}X_{12}-qX_{22}X_{11})+X_{11}X_{22}\otimes(X_{11}X_{22}-qX_{12}X_{21})\\ &= X_{12}X_{21}\otimes(q^2X_{12}X_{21}-qX_{11}X_{22})+X_{11}X_{22}\otimes 1\\ &= -qX_{12}X_{21}\otimes(X_{11}X_{22}-qX_{12}X_{21})+X_{11}X_{22}\otimes 1\\ &= X_{11}X_{22}-qX_{12}X_{21}\otimes 1\\ &= 1\otimes 1\end{aligned}$$

Thus $\Delta(D_q-1)=0$ and so, we have a k-algebra homomorphism

$$\Delta:\mathcal{O}_q(Sl_2(k))\longrightarrow\mathcal{O}_q(Sl_2(k))\otimes\mathcal{O}_q(Sl_2(k)).$$

Recall that the counit $\varepsilon:\mathcal{O}_q(\mathrm{Mat}_2(k))\longrightarrow k$ is given by $\varepsilon(X_{ij})=\delta_{ij}$. Then,

$$\varepsilon(D_q)=\varepsilon(X_{11}X_{22}-qX_{12}X_{21})=\varepsilon(X_{11}X_{22})-\varepsilon(qX_{12}X_{21})=1.$$

Thus $\varepsilon(D_q-1)=0$, and so, we have a k-algebra homomorphism

$$\varepsilon:\mathcal{O}_q(Sl_2(k))\longrightarrow k.$$

This implies that $(\mathcal{O}_q(Sl_2(k)),\Delta,\varepsilon)$ is a bialgebra.

Given an algebra A, *the opposite algebra* A^{op} is the algebra such that as k-vector space is equal to A and the product is given by $a\cdot b=ba\in A$.

Consider the following map $S:k\langle X_{11},X_{12},X_{21},X_{22}\rangle\longrightarrow\mathcal{O}_q(Sl_2(k))^{op}$ defined in generators as

$$\begin{array}{ll} S(X_{11})=X_{22} & S(X_{12})=-q^{-1}X_{12}\\ S(X_{21})=-qX_{21} & S(X_{22})=X_{11}.\end{array}$$

We have to check that this k-algebra homomorphism can be defined from $\mathcal{O}_q(Sl_q(k))$, that is, we have to check that S respects the relations 2.1,2.3, 2.4 and $S(D_q)=1$.

$$\begin{aligned}S(X_{12}X_{22}) &= S(X_22)S(X_{12})\\ &= -q^{-1}X_{11}X_{12}\\ &= -X_{12}X_{11}\\ &= S(qX_{22}X_{12}).\end{aligned}$$

$$\begin{aligned} S(X_{12}X_{21}) &= S(X_{21})S(X_{12}) \\ &= (-qX_{21})(-q^{-1}X_{12}) \\ &= X_{21}X_{12} \\ &= X_{12}X_{21} \\ &= (-q^{-1}X_{12})(-qX_{21}) \\ &= S(X_{21}X_{12}) \end{aligned}$$

$$\begin{aligned} S(X_{11}X_{22} - X_{22}X_{11}) &= S(X_{11}X_{22}) - S(X_{22}X_{11}) \\ &= X_{11}X_{22} - X_{22}X_{11} \\ &= (q - q^{-1})X_{12}X_{21} \\ &= (q - q^{-1})S(X_{12}X_{21}). \end{aligned}$$

$$\begin{aligned} S(D_q) &= S(X_{11}X_{22} - qX_{12}X_{21}) \\ &= X_{11}X_{22} - X_{12}X_{21} \\ &= 1 \end{aligned}$$

Hence we have an anti-automorphism

$$S : \mathcal{O}_q(Sl_2(k)) \longrightarrow \mathcal{O}_q(Sl_q(k)).$$

Now, note that

$$\begin{aligned} (id * S)(X_{11}) &= X_{11}S(X_{11}) + X_{12}S(X_{21}) \\ &= X_{11}X_{22} - qX_{12}X_{21} \\ &= 1 \\ &= u\varepsilon(X_{11}) \end{aligned}$$

$$\begin{aligned} (id * S)(X_{12}) &= X_{11}S(X_{12}) + X_{12}S(X_{22}) \\ &= -q^{-1}X_{11}X_{12} + X_{12}X_{11} \\ &= 0 \\ &= u\varepsilon(X_{12}) \end{aligned}$$

Thus, $(\mathcal{O}_q(Sl_2(k)), \Delta, \varepsilon, S)$ is a Hopf algebra.

Exercise I.1.I

Exercise I.1.J

Consider the quantum plane

$$\mathcal{O}_q(k^2) = k\langle x_1, x_2\rangle / \langle x_2x_1 - q^{-1}x_1x_2\rangle.$$

Hence, the reduction system is just

$$\{(x_2x_1, q^{-1}x_1x_2)\}.$$

That implies that there are no inclusion ambiguities and overlap ambiguities. Note that the irreducible monomials are $x_1^i x_2^j$ for all $i, j > 0$. Thus, by the Diamond Lemma ([1, I.11.6]), $\{x_1^i x_2^j \mid i, j \geq 0\}$ is a basis for the quantum plane.

Exercise I.1.K

Using the Diamond Lemma, it can be seen that the monomials $a^\bullet b^\bullet c^\bullet d^\bullet$ in $\mathcal{O}(\mathrm{Mat}_2(k))$ are linearly independent ([1, I.11.7]). We claim that d is a regular element of $\mathcal{O}_q(\mathrm{Mat}_2(k))$. Let $x \in \mathcal{O}_q(\mathrm{Mat}_2(k))$ such that $xd = 0$. We can write $x = \sum_{i=1}^n k_i a^{\ell_i} b^{m_i} c^{n_i} d^{j_i}$. Then

$$0 = xd = \sum_{i=1}^n k_i a^{\ell_i} b^{m_i} c^{n_i} d^{j_i} d = \sum_{i=1}^n k_i a^{\ell_i} b^{m_i} c^{n_i} d^{j_i+1}.$$

This implies that $k_i = 0$ for $1 \leq i \leq n$. Therefore $x = 0$.

Now suppose $dx = 0$, that is,

$$0 = d \sum_{i=1}^n k_i a^{\ell_i} b^{m_i} c^{n_i} d^{j_i} = \sum_{i=1}^n k_i d a^{\ell_i} b^{m_i} c^{n_i} d^{j_i}.$$

Let us look at the term da^{ℓ_i} and suppose $\ell_i > 0$. Then

$$\begin{aligned} da^{\ell_i} &= (da)a^{\ell_i - 1} \\ &= (ad - \widehat{q}bc)a^{\ell_i-1} \\ &= ada^{\ell_i-1} - \widehat{q}bca^{\ell_i-1} \\ &= ada^{\ell_i-1} - \widehat{q}bq^{-(\ell_i-1)}a^{\ell_i-1}c \\ &= ada^{\ell_i-1} - \widehat{q}q^{-2(\ell_i-1)}a^{\ell_i-1}bc \end{aligned}$$

Therefore

$$da^{\ell_i} = a^{\ell_i} d - \sum_{h=1}^{\ell_i} \widehat{q}q^{-2(\ell_i-h)}a^{\ell_i-h}bc.$$

Hence,

$$\begin{aligned} 0 &= \sum_{i=1}^n k_i d a^{\ell_i} b^{m_i} c^{n_i} d^{j_i} \\ &= \sum_{i=1}^n k_i \left(a^{\ell_i} d - \sum_{h=1}^{\ell_i} \widehat{q}q^{-2(\ell_i-h)}a^{\ell_i-h}bc \right) b^{m_i} c^{n_i} d^{j_i} \\ &= \sum_{i=1}^n k_i a^{\ell_i} d b^{m_i} c^{n_i} d^{j_i} - \sum_{h=1}^{\ell_i} k_i \widehat{q}q^{-2(\ell_i-h)}a^{\ell_i-h}bcb^{m_i} c^{n_i} d^{j_i} \\ &= \sum_{i=1}^n k_i a^{\ell_i} q^{-m_i} q^{-n_i} b^{m_i} c^{n_i} d d^{j_i} - \sum_{h=1}^{\ell_i} k_i \widehat{q}q^{-2(\ell_i-h)}a^{\ell_i-h}b^{m_i+1} c^{n_i+1} d^{j_i} \\ &= \sum_{i=1}^n k_i q^{-(m_i+n_i)} a^{\ell_i} b^{m_i} c^{n_i} d^{j_i+1} - \sum_{h=1}^{\ell_i} k_i \widehat{q}q^{-2(\ell_i-h)}a^{\ell_i-h}b^{m_i+1} c^{n_i+1} d^{j_i} \end{aligned}$$

Note that all the monomials are different and since they are linearly independent, we have that $0 = k_i q^{-(m_i+n_i)}$ for all $1 \leq i \leq n$ which implies that $k_i = 0$. Thus $x = 0$ and so d is a regular element.

Note that with a similar argument we have that $dA \cap A = 0 = Ad \cap A$ where A is the k-subalgebra of $\mathcal{O}_q(\mathrm{Mat}_2(k^2))$ generated by a, b, c.

Exercise I.1.L

Let $R \subseteq S$ be rings, and suppose that there is a regular element $d \in S$ such that $dR + R = Rd + R$ and $dR \cap R = 0 = Rd \cap R$. Then there are unique maps $\tau, \delta : R \longrightarrow R$ such that $dr = \tau(r)d + \delta(r)$ for all $r \in R$. Show that τ is an automorphism of R and that δ is a τ-derivation on R.

Since $dR + R = Rd + R$, for $r \in R$, let $\tau(r)$ and $\delta(r)$ elements in R such that $dr = \tau(r)d + \delta(r)$. If $dr = ad + b = cd + e$, then $(a - c)d = e - b \in Rd \cap R = 0$. Hence $(a-c)d = 0$. Since d is regular, $a = c$, and so $b = e$. Therefore, $\tau(r)$ and $\delta(r)$ are unique and so we have functions $\tau, \delta : R \longrightarrow R$. Analogously, there exist τ' and δ' such that $rd = d\tau'(r) + \delta'(r)$. Let $r, t \in R$. Then $d(r + t) = \tau(r + t)d + \delta(r + t)$. On the other hand,

$$d(r + t) = dr + dt = \tau(r)d + \delta(r) + \tau(t)d + \delta(t) = (\tau(r) + \tau(t))d + \delta(r) + \delta(t).$$

It follows that $\tau(r + t) = \tau(r) + \tau(t)$ and $\delta(r + t) = \delta(r) + \delta(t)$. Note that, if $\tau(r) = 0$, then $dr = \delta(r) \in dR \cap R = 0$. Hence $dr = 0$ and so $r = 0$. Thus, τ and τ' are injective. Now, we have that

$$\tau(r)d = d\tau'(\tau(r)) + \delta'(\tau(r)) = \tau\tau'\tau(r)d + \delta(\tau'\tau(r)) + \delta'(\tau(r)).$$

Hence $\tau(r) = \tau\tau'\tau(r)$. It follows that $\tau'\tau(r) = r$. Analogously, $\tau\tau'(r) = r$. On the other hand,

$$\begin{aligned} drs &= \tau(r)ds + \delta(r)s = \tau(r)(\tau(s)d + \delta(s)) + \delta(r)s \\ &= \tau(r)\tau(s)d + \tau(r)\delta(s) + \delta(r)s. \\ drs &= \tau(rs)d + \delta(rs). \end{aligned}$$

Hence, $\tau(rs) = \tau(r)\tau(s)$ and $\delta(rs) = \tau(r)\delta(s) + \delta(r)s$. Thus, τ is an automorphism of R and δ is a τ-derivation on R.

Exercise I.1.M

Let $F = k\langle X_1, ..., X_t\rangle$ be the free algebra over k on letters $X_1, ..., X_t$ and let τ be a k-algebra endomorphism of F. Given any $f_1, ..., f_t \in F$ show that there exists a unique k-linear τ-derivation δ on F such that $\delta(X_i) = f_i$ for all $1 \leq i \leq t$.

Let $\phi : F \longrightarrow \mathrm{Mat}_2(F)$ be the k-algebra homomorphism given by $\phi(X_i) = \left(\begin{smallmatrix} \tau(X_i) & f_i \\ 0 X_i \end{smallmatrix}\right)$. Then

$$\phi(X_iX_j) = \phi(X_i)\phi(X_j) = \begin{pmatrix} \tau(X_i) & f_i \\ 0 & X_i \end{pmatrix}\begin{pmatrix} \tau(X_j) & f_j \\ 0 & X_j \end{pmatrix} = \begin{pmatrix} \tau(X_i)\tau(X_j) & \tau(X_i)f_j + f_iX_j \\ 0 & X_iX_j \end{pmatrix}.$$

So, define $\delta : F \longrightarrow F$ as $\delta(X) = (1\ 0)\phi(X)\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for all $X \in F$. Then $\delta(X_iX_j) = \tau(X_i)f_j + f_iX_j$. Consider X_1 and $X_{i(1)} \cdots X_{i(t)}$ any monomial. We are going to prove that δ is a τ-derivation on F by induction on t. Then,

$$\begin{aligned} &\delta(X_1X_{i(1)} \cdots X_{i(t)}) \\ &= (1\ 0)\phi(X_1X_{i(1)} \cdots X_{i(t)})\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \\ &= (1\ 0)\phi(X_1)\phi(X_{i(1)} \cdots X_{i(t)})\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \\ &= (1\ 0)\begin{pmatrix} \tau(X_1) & f_1 \\ 0 & X_1 \end{pmatrix}\begin{pmatrix} \tau(X_{i(1)} \cdots X_{i(t)}) & \tau(X_{i(1)})\delta(X_{i(2)} \cdots X_{i(t)}) + \delta(X_{i(1)})X_{i(2)} \cdots X_{i(t)} \\ 0 & X_{i(1)} \cdots X_{i(t)} \end{pmatrix}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \\ &= \tau(X_1)(\tau(X_{i(1)})\delta(X_{i(2)} \cdots X_{i(t)}) + \delta(X_{i(1)})X_{i(2)} \cdots X_{i(t)}) + f_1X_{i(1)} \cdots X_{i(t)} \\ &= \tau(X_1)\delta(X_{i(1)} \cdots X_{i(t)}) + \delta(X_1)X_{i(1)} \cdots X_{i(t)}. \end{aligned}$$

Hence δ is a τ-derivation on F, and it is clear that δ is unique.

Now let $I = \langle G\rangle$ be the ideal generated by some set $G \subseteq F$. If $\tau(g), \delta(g) \in I$ for all $g \in G$, show that I is stable under τ and δ. Let $r, s \in F$ and $g \in G$. Then

$\tau(rgs) = \tau(r)\tau(g)\tau(s)$. By hypothesis, $\tau(g) \in I$, hence $\tau(rgs) \in I$. On the other hand,

$$\begin{aligned}\delta(rgs) &= \tau(r)\delta(gs) + \delta(r)gs \\ &= \tau(r)(\tau(g)\delta(s) + \delta(g)s) + \delta(r)gs \\ &= \tau(r)\tau(g)\delta(s) + \tau(r)\delta(g)s + \delta(r)gs.\end{aligned}$$

Since $\tau(g), \delta(g), g \in I$, $\delta(rgs) \in I$. It follows that τ induces a k-algebra endomoephism of F/I and δ induces a $\overline{\tau}$-derivation on F/I.

Exercise I.1.M

Consider the algebra $\mathcal{O}_q(\mathrm{Mat}_2(k))$. We want to use the model approach to see that this algebra is isomorphic to an iterated skew polynomial ring. First, we construct an iterated skew polynomial algebra

$$B = k[x][y;\sigma_2][z;\sigma_3]$$

where $k[x]$ is a polynomial ring, σ_2 is the k-algebra automorphism of $k[x]$ such that $\sigma_2(x) = q^{-1}x$, and σ_3 is the k-algebra automorphism of $k[x][y;\sigma_2]$ such that $\sigma_3(x) = q^{-1}x$ and $\sigma_3(y) = y$. Consider the k-algebra homomorphism $\sigma_4 : k\langle x,y,z\rangle \longrightarrow k\langle x,y,z\rangle$ given by $\sigma_4(x) = x$, $\sigma_4(y) = q^{-1}y$ and $\sigma_4(z) = q^{-1}z$. By exercise I.1.L there exists a unique σ_4-derivation δ_4 on $k\langle x,y,z\rangle$ such that $\delta_4(x) = (q^{-1} - q)yz$ and $\delta_4(y) = 0 = \delta_4(z)$. Consider the ideal

$$I = \left\langle yx - q^{-1}xy, zx - q^{-1}xz, zy - yz \right\rangle.$$

Hence $B = k\langle x,y,z\rangle / I$. Let see that the images of the generators of I under δ_4 are in I.

$$\begin{aligned}\delta_4(yx) &= \sigma_4(y)\delta_4(x) + \delta_4(y)x \\ &= q^{-1}y(q^{-1} - q)yz \\ &= (q^{-2} - 1)yyz.\delta(xy) = \sigma_4(x)\delta_4(y) + \delta_4(x)y = (q^{-1} - q)yzy.\end{aligned}$$

Then,

$$\delta_4(yx - q^{-1}xy) = (q^{-2} - 1)yyz - q^{-1}(q^{-1} - q)yzy = (q^{-2} - 1)y(yz - zy).$$

We also have that,

$$\delta_4(zy) = \sigma_4(z)\delta_4(y) + \delta_4(z)y = 0 = \delta_4(yz).$$

And,

$$\begin{aligned}\delta_4(zx) &= \sigma_4(z)\delta_4(x) + \delta_4(z)x \\ &= q^{-1}z(q^{-1} - q)yz \\ &= (q^{-2} - 1)zyz. \\ \delta_4(xz) &= \sigma_4(x)\delta_4(z) + \delta_4(x)z \\ &= (q^{-1} - q)yzz.\end{aligned}$$

Then

$$\delta_4(zx - q^{-1}xz) = (q^{-2} - 1)zyz - q^{-1}(q^{-1} - q)yzz = (q^{-2} - 1)(zy - yz)z.$$

For the automorphism σ_4, we have that

$$\sigma_4(yx) = q^{-1}yx$$

$$\sigma_4(xy) = x(q^{-1}y)$$

Then,

$$\sigma_4(yx - q^{-1}xy) = q^{-1}yx - q^{-2}xy = q^{-1}(yx - q^{-1}xy).$$

Also,

$$\sigma_4(zx) = q^{-1}zx$$
$$\sigma_4(xz) = x(q^{-1}z)$$

Then,

$$\sigma_4(zx - q^{-1}xz) = q^{-1}zx - q^{-2}xz = q^{-1}(zx - q^{-1}xz)$$

And,

$$\sigma_4(zy) = q^{-1}zq^{-1}y = q^{-2}zy$$
$$\sigma_4(yz) = q^{-1}yq^{-1}z = q^{-2}yz$$

Then,

$$\sigma_4(zy - yz) = q^{-2}zy - q^{-2}yz = q^{-2}(yz - zy).$$

Thus, σ_4 induces an automorphism of B and δ_4 induces a σ_4-derivation on B.

Exercise I.1.O

Construct a k-algebra isomorphism of $\mathcal{O}_q(GL_2(k))$ onto Laurent polynomial ring $\mathcal{O}_q(Sl_2(k))[z^{\pm}]$. Consider the k-algebra homomorphism

$$\phi : k\langle X_{11}, X_{12}, X_{21}, X_{22}\rangle \longrightarrow \mathcal{O}_q(Sl_2(k))[z^{\pm}]$$

given by $\phi(X_{11}) = \overline{X_{11}}z$, $\phi(X_{12}) = \overline{X_{12}}z$, $\phi(X_{21}) = \overline{X_{21}}$ and $\phi(X_{22}) = \overline{X_{22}}$. We have to check that ϕ preserves the relations 2.1,2.3,2.4.

$$\begin{aligned}
\phi(X_{11}X_{21}) &= \phi(X_{11})\phi(X_{21}) \\
&= \overline{X_{11}}z\overline{X_{21}} \\
&= z\overline{X_{11}X_{21}} \\
&= qz\overline{X_{21}X_{11}}. \\
q\phi(X_{21}X_{11}) &= q\phi(X_{21})\phi(X_{11}) \\
&= q\overline{X_{21}X_{11}}z \\
&= qz\overline{X_{21}X_{11}}.
\end{aligned}$$

Analogously the other relations in 2.1 are preserved. For 2.3,

$$\begin{aligned}
\phi(X_{12}X_{21}) &= \phi(X_{12}X_{21}) \\
&= \overline{X_{12}}z\overline{X_{21}} \\
&= \overline{X_{21}X_{12}}z \\
&= \phi(X_{21})\phi(X_{12}) \\
&= \phi(X_{21}X_{12}).
\end{aligned}$$

For 2.4,

$$\begin{aligned}
\phi(X_{11}X_{22} - X_{22}X_{11}) &= \phi(X_{11})\phi(X_{22}) - \phi(X_{22})\phi(X_{11}) \\
&= \overline{X_{11}}z\overline{X_{22}} - \overline{X_{22}X_{11}}z \\
&= (\overline{X_{11}X_{22}} - \overline{X_{22}X_{11}})z \\
&= (q - q^{-1})\overline{X_{21}X_{12}}z \\
&= (q - q^{-1})\phi(X_{21})\phi(X_{12}) \\
&= \phi((q - q^{-1})X_{21}X_{12}).
\end{aligned}$$

Thus, there is a k-algebra homomorphism

$$\widehat{\phi}:\mathcal{O}_q(\mathrm{Mat}_2(k))\longrightarrow \mathcal{O}_q(Sl_2(k))[z^{\pm}].$$

Note that,

$$\begin{aligned}\widehat{\phi}(D_q) &= \widehat{\phi}(X_{11}X_{22}-qX_{12}X_{21})\\ &= \widehat{\phi}(X_{11})\widehat{\phi}(X_{22})-q\widehat{\phi}(X_{12})\widehat{\phi}(X_{21})\\ &= z\overline{X_{11}X_{22}}-qz\overline{X_{12}X_{21}}\\ &= z(\overline{X_{11}X_{22}}-q\overline{X_{12}X_{21}})\\ &= z.\end{aligned}$$

Hence $\widehat{\phi}$ sends D_q to an invertible element and so there exists a k-algebra homomorphism

$$\overline{\phi}:\mathcal{O}_q(GL_2(k))\longrightarrow \mathcal{O}_q(Sl_2(k))[z^{\pm}].$$

In order to see that $\overline{\phi}$ is an isomorphism, we will give its inverse.

Define $\psi : k\langle X_{11},X_{12},X_{21},X_{22}\rangle \longrightarrow \mathcal{O}_q(GL_n(k))$ as $\psi(X_{11})=\frac{X_{11}}{D_q}$, $\psi(X_{12})=\frac{X_{12}}{D_q}$, $\psi(X_{21})=X_{21}$ and $\psi(X_{22})=X_{22}$. It is not difficult to see that ψ induces a k-algebra homomorphism

$$\widehat{\psi}:\mathcal{O}_q(\mathrm{Mat}_2(k))\longrightarrow \mathcal{O}_q(GL_2(k)).$$

Then,

$$\begin{aligned}\widehat{\psi}(D_q) &= \widehat{\psi}(X_{11}X_{22}-qX_{12}X_{21})\\ &= \widehat{\psi}(X_{11})\widehat{\psi}(X_{22})-q\widehat{\psi}(X_{12})\widehat{\psi}(X_{21})\\ &= \frac{X_{11}}{D_q}X_{22}-q\frac{X_{12}}{D_q}X_{21}\\ &= \frac{X_{11}X_{22}-qX_{12}X_{21}}{D_q}\\ &= 1.\end{aligned}$$

Thus, $\widehat{\psi}$ induces a k-algebra homomorphism

$$\widehat{\psi}':\mathcal{O}_q(Sl_2(k))\longrightarrow \mathcal{O}_q(GL_2(k)).$$

Hence, there exists a k-algebra homomorphism

$$\overline{\psi}:\mathcal{O}_q(Sl_2(k))[z^{\pm}]\longrightarrow \mathcal{O}_q(GL_2(k))$$

sending $\overline{\psi}(z)=D_q$. It is clear that $\overline{\psi}$ is the inverse of $\overline{\phi}$.

Exercise I.3.A

Let $r,s\in R$. Then

$$\begin{aligned}\phi(r)\phi(s) &= \begin{pmatrix}\alpha(r) & \delta(r)\\ 0 & r\end{pmatrix}\begin{pmatrix}\alpha(s) & \delta(s)\\ 0 & s\end{pmatrix}\\ &= \begin{pmatrix}\alpha(r)\alpha(s) & \alpha(r)\delta(s)+\delta(r)s\\ 0 & rs\end{pmatrix}\\ \phi(rs) &= \begin{pmatrix}\alpha(rs) & \delta(rs)\\ 0 & rs\end{pmatrix}\end{aligned}$$

It is clear that $\phi(rs)=\phi(r)\phi(s)$ if and only if δ is an α-derivation.

Ch. I.3 Proposition

Proof. Let $\tau_1 : k[K,K^{-1}] \longrightarrow k[K,K^{-1}]$ be the automorphism given by $\tau_1(K) = q^{-2}K$. Then we can construct the skew polynomial ring $k[K,K^{-1}][E;\tau_1]$. Note that,

$$KEK^{-1} = K\tau_1(K^{-1})E = Kq^2K^{-1}E = q^2E.$$

Let $\tau : k[K,K^{-1}] \longrightarrow k[K,K^{-1}][E;\tau_1]$ given by $\tau(K) = q^2K$. Then

$$E\tau(K) = Eq^2K = KE = \tau(\tau_1(K))E$$

$$E\tau(K^{-1}) = Eq^{-2}K^{-1} = K^{-1}E = \tau(\tau_1(K^{-1}))E.$$

By the universal property of skew polynomial rings [6, 2.5], there exists a unique automorphism

$$\tau_2 : k[K,K^{-1}][E;\tau_1] \longrightarrow k[K,K^{-1}][E;\tau_1]$$

sending $\tau_2(K) = q^2K$ and $\tau_2(E) = E$. Consider the following ring homomorphism

$$\phi : k\left\langle K,K^{-1},E\right\rangle \longrightarrow \mathrm{Mat}_2(k[K,K^{-1}][E;\tau_1])$$

given by $\phi(K) = \begin{pmatrix} q^2K & 0 \\ 0 & K \end{pmatrix}$ and $\phi(E) = \begin{pmatrix} E & \frac{K^{-1}-K}{q-q^{-1}} \\ 0 & E \end{pmatrix}$. Hence

$$\begin{aligned} \phi(\tau_1(K)E) &= \phi(q^{-2}KE) \\ &= q^{-2}\begin{pmatrix} q^2K & 0 \\ 0 & K \end{pmatrix}\begin{pmatrix} E & \frac{K^{-1}-K}{q-q^{-1}} \\ 0 & E \end{pmatrix} \\ &= \begin{pmatrix} KE & \frac{1-K^2}{q-q^{-1}} \\ 0 & q^{-2}KE \end{pmatrix} \\ \phi(EK) &= \begin{pmatrix} E & \frac{K^{-1}-K}{q-q^{-1}} \\ 0 & E \end{pmatrix}\begin{pmatrix} q^2K & 0 \\ 0 & K \end{pmatrix} \\ &= \begin{pmatrix} q^2EK & \frac{1-K^2}{q-q^{-1}} \\ 0 & EK \end{pmatrix} \\ &= \begin{pmatrix} KE & \frac{1-K^2}{q-q^{-1}} \\ 0 & q^{-2}KE \end{pmatrix} \end{aligned}$$

Thus ϕ induces a ring homomorphism $\overline{\phi} : k[K,K^{-1}][E;\tau_1] \longrightarrow \mathrm{Mat}_2(k[K,K^{-1}][E;\tau_1])$. By exercise I.3.A there exists a unique τ_2-derivation δ_2 on $k[K,K^{-1}][E;\tau_1]$ such that $\delta_2(K) = 0$ and $\delta_2(E) = \frac{K^{-1}-K}{q-q^{-1}}$. So we have an iterated skew polynomial ring $A = k[K,K^{-1}][E;\tau_1][F;\tau_2,\delta_2]$. To see the isomorphism, we only have to check that the defining relations in $\mathcal{U}_q(\mathfrak{sl}_2(k))$ are valid in A. For,

$$KEK^{-1} = K\tau_1(K^{-1})E = Kq^2K^{-1}E = q^2E.$$

$$KFK^{-1} = K(\tau_2(K^{-1})F + \delta_2(K^{-1})) = Kq^{-2}K^{-1}F = q^{-2}F.$$

$$FE = \tau_2(E)F + \delta_2(E) = EF + \frac{K^{-1}-K}{q-q^{-1}}$$

then $EF - FE = \frac{K^{-1}-K}{q-q^{-1}}$. Thus, $A \cong \mathcal{U}_q(\mathfrak{sl}_2(k))$. □

Hopf algebra structure of $\mathcal{U}_q(\mathfrak{sl}_2(k))$

We have that

$$\mathcal{U}_q(\mathfrak{sl}_2(k)) = \frac{k\left\langle E,F,K,K^{-1}\right\rangle}{\left\langle KEK^{-1}-q^2E, KFK^{-1}-q^{-2}F, EF-FE-\frac{(K-K^{-1})}{q-q^{-1}}\right\rangle}$$

There is a k-algebra homomorphism

$$\Delta : k\left\langle E,F,K,K^{-1}\right\rangle \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2(k)) \otimes \mathcal{U}_q(\mathfrak{sl}_2(k))$$

sending $\Delta(E) = E \otimes 1 + K \otimes E$, $\Delta(F) = F \otimes K^{-1} + 1 \otimes F$ and $\Delta(K) = K \otimes K$.

Let us check that Δ preserves the relations.

$$\begin{aligned}\Delta(KEK^{-1}) &= \Delta(K)\Delta(E)\Delta(K^{-1})\\ &= (K \otimes K)(E \otimes 1 + K \otimes E)(K^{-1} \otimes K^{-1})\\ &= (KE \otimes K + K^2 \otimes KE)(K^{-1} \otimes K^{-1})\\ &= KEK^{-1} \otimes 1 + K \otimes KEK^{-1}\\ &= -q^2E \otimes 1 + K \otimes -q^2E\\ &= -q^2(E \otimes 1 + K \otimes E)\\ &= \Delta(-q^2E)\end{aligned}$$

$$\begin{aligned}\Delta(KFK^{-1}) &= \Delta(K)\Delta(F)\Delta(K^{-1})\\ &= (K \otimes K)(F \otimes K^{-1} + 1 \otimes F)(K^{-1} \otimes K^{-1})\\ &= (KF \otimes 1 + K \otimes KF)(K^{-1} \otimes K^{-1})\\ &= KFK^{-1} \otimes K^{-1} + 1 \otimes KFK^{-1}\\ &= q^{-2}F \otimes K^{-1} + 1 \otimes q^{-2}F\\ &= q^{-2}(F \otimes K^{-1} + 1 \otimes F)\\ &= \Delta(q^{-2}F)\end{aligned}$$

$$\begin{aligned}\Delta(EF) &= (E \otimes 1 + K \otimes E)(F \otimes K^{-1} + 1 \otimes F)\\ &= EF \otimes K^{-1} + E \otimes F + KF \otimes EK^{-1} + K \otimes EF\end{aligned}$$

$$\Delta(FE) = FE \otimes K^{-1} + E \otimes F + FK \otimes K^{-1}E + K \otimes FE$$

Then,

$$\begin{aligned}\Delta(EF - FE) &= (EF - FE) \otimes K^{-1} + K \otimes (EF - FE) + q^{-2}FK \otimes EK^{-1} - FK \otimes K^{-1}E\\ &= \frac{(K - K^{-1})}{q - q^{-1}} \otimes K^{-1} + K \otimes \frac{(K - K^{-1})}{q - q^{-1}} + FK \otimes (q^{-2}EK^{-1} - K^{-1}E)\\ &= \frac{(K - K^{-1})}{q - q^{-1}} \otimes K^{-1} + K \otimes \frac{(K - K^{-1})}{q - q^{-1}} + FK \otimes (K^{-1}E - K^{-1}E)\\ &= \frac{K \otimes K^{-1} - K^{-1} \otimes K^{-1} + K \otimes K - K \otimes K^{-1}}{q - q^{-1}}\\ &= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}}\\ &= \Delta\left(\frac{K - K^{-1}}{q - q^{-1}}\right)\end{aligned}$$

So, there is a unique k-algebra homomorphism

$$\Delta : \mathcal{U}_q(\mathfrak{sl}_2(k)) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2(k)) \otimes \mathcal{U}_q(\mathfrak{sl}_2(k))$$

To see that Δ is coassociative, it just has to be proven in the generators.

Consider the k-algebra homomorphism

$$S : k\left\langle E, F, K, K^{-1}\right\rangle \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2(k))^{op}$$

given by $S(K) = K^{-1}$, $S(E) = -K^{-1}E$ and $S(F) = -FK$. Then,

$$\begin{aligned}
S(KEK^{-1}) &= S(K^{-1})S(KE) \\
&= S(K^{-1})S(E)S(K) \\
&= K(-K^{-1}E)K^{-1} \\
&= -EK^{-1} \\
&= -q^2K^{-1}E \\
S(q^2E) &= q^2S(E) \\
&= -q^2K^{-1}E
\end{aligned}$$

$$\begin{aligned}
S(KFK^{-1}) &= S(K^{-1})S(F)S(K) \\
&= K(-FK)K^{-1} \\
&= -KF \\
&= -q^{-2}FK \\
S(q^{-2}F) &= -q^{-2}FK
\end{aligned}$$

$$S(EF) = (-FK)(-K^{-1}E) = FE$$
$$S(FE) = (-K^{-1}E)(-FK) = K^{-1}EFK$$

so,

$$\begin{aligned}
S(EF - FE) &= FE - K^{-1}EFK \\
&= FE - (q^{-2}EK^{-1})(q^2KF) \\
&= FE - EF \\
&= \frac{K^{-1} - K}{q - q^{-1}} \\
&= S\left(\frac{K - K^{-1}}{q - q^{-1}}\right)
\end{aligned}$$

Therefore S induces a unique k-algebra anti-homomorphism

$$S : \mathcal{U}_q(\mathfrak{sl}_2(k)) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2(k)).$$

Note that

$$S^2(K) = S(K^{-1}) = K = K^{-1}KK$$
$$S^2(E) = S(-K^{-1}E) = -S(E)S(K^{-1}) = K^{-1}EK$$
$$S^2(F) = S(-FK) = -K^{-1}(-FK) = K^{-1}FK$$

This implies that S is bijetive and hence so is S. Now, let us prove that S is an inverse of Id with the convolution product.

$$\begin{aligned}
(S * Id)(K) &= S(K)K \\
&= 1 \\
&= \varepsilon(K)
\end{aligned}$$

$$\begin{aligned}(S*Id)(E) &= S(E)+S(K)E\\ &= -K^{-1}E+K^{-1}E\\ &= 0\\ &= \varepsilon(E)\end{aligned}$$

$$\begin{aligned}(S*Id)(F) &= S(F)K^{-1}+F\\ &= -FKK^{-1}+F\\ &= -F+F\\ &= 0\\ &= \varepsilon(F)\end{aligned}$$

Lemma 2.2. *Let H be a Hopf algebra. Let $a,b\in H$ such that $(S*Id)(a)=\varepsilon(a)$ and $(S*Id)(b)=\varepsilon(b)$. Then $(S*Id)(ab)=\varepsilon(ab)$.*

Proof.

$$\begin{aligned}(S*Id)(ab) &= \mu\circ(S\otimes Id)\circ\Delta(ab)\\ &= \mu\circ(S\otimes Id)(\Delta(a)\Delta(b))\\ &= \mu\circ(S\otimes Id)(\sum a_1b_1\otimes a_2b_2)\\ &= \sum S(a_1b_1)a_2b_2\\ &= \sum S(b_1)S(a_1)a_2b_2\\ &= \sum S(b_1)\varepsilon(a)b_2\\ &= \varepsilon(a)\sum S(b_1)b_2\\ &= \varepsilon(a)\varepsilon(b)\end{aligned}$$

□

By the lemma S is an antipode for $\mathcal{U}_q(\mathfrak{sl}_2(k))$. Therefore $\mathcal{U}_q(\mathfrak{sl}_2(k))$ is a Hopf algebra.

Lemma 2.3. *Consider the quantum plane $\mathcal{O}_q(k^2)$. Then, the multiplicative set X in $\mathcal{O}_q(k^2)$ generated by x and y is a denominator set.*

Proof. Let $\sum f_i(x)y^i\in\mathcal{O}_q(k^2)$ and $q^\ell x^m y^n\in X$. Then

$$\begin{aligned}\sum f_i(x)y^i(q^\ell x^m y^n) &= q^\ell\sum f_i(x)(q^{-im}x^m y^i)y^n\\ &= q^\ell x^m\sum q^{-im}f_i(x)y^n y^i\end{aligned}$$

Suppose that $f_i(x)=a_{i_k}x^k+a_{i_{k-1}}x^{k-1}+\cdots+a_{i_1}x+a_{i_0}$. Then

$$\begin{aligned}(a_{i_k}x^k+a_{i_{k-1}}x^{k-1}+\cdots+a_{i_1}x+a_{i_0})y^n &= a_{i_k}x^k y^n+a_{i_{k-1}}x^{k-1}y^n+\cdots+a_{i_1}xy^n+a_{i_0}y^n\\ &= q^{kn}y^n a_{i_k}x^k+q^{(k-1)n}y^n a_{i_{k-1}}x^{k-1}+\cdots+q^n y^n a_{i_1}x+y^n a_{i_0}\end{aligned}$$

For each i, define

$$g_i(x)=q^{kn}a_{i_k}x^k+q^{(k-1)n}a_{i_{k-1}}x^{k-1}+\cdots+q^n a_{i_1}x+a_{i_0}$$

Then

$$(\sum f_i(x)y^i)q^\ell x^m y^n = (q^\ell x^m y^n)\sum q^{-im} g_i(x)y^i.$$

Thus, X is a left denominator set. Analogously, X is a right denominator set. $\square$

Exercise II.1.B

We claim that $\mathcal{O}_q((k^x)^2) = k\langle x, x^{-1}, y, y^{-1} \mid xy = qyx\rangle$ is a simple ring. Let I be a nonzero ideal of $\mathcal{O}_q((k^x)^2)$ and let $0 \neq a \in I$. Since I is an ideal, we can assume $a = \sum_{i=0}^n f_i(x)y^i \in \mathcal{O}_q(k^2)$. By induction on n, we will prove that I contains a unit.

$n = 0$. Then $a = f_0(x) = b_m x^m + \cdots + b_1 x + b_0$. By induction on m. If $m = 0$, $a = b_0 \in k$. Now, suppose that $m > 0$. Then

$$\begin{aligned} ya &= y(b_m x^m + \cdots + b_1 x + b_0) \\ &= b_m y x^m + \cdots + b_1 yx + b_0 y \\ &= q^{-m} b_m x^m y + \cdots + q^{-1} b_1 xy + b_0 y \end{aligned}$$

Since I is an ideal $ya - q^{-m}ay \in I$, and so

$$ya - q^{-m}ay = (q^{-(m-1)} - q^{-m})b_{m-1}x^{m-1}y + \cdots + (q^{-1} - q^{-m})b_1 xy + (1 - q^{-m})b_0 y \in I$$

Since y is invertible,

$$(q^{-(m-1)} - q^{-m})b_{m-1}x^{m-1} + \cdots + (q^{-1} - q^{-m})b_1 x + (1 - q^{-m})b_0 \in I.$$

By induction hypothesis, I contains a unit.

Now suppose $n > 0$. Then

$$\begin{aligned} ax &= f_n(x)y^n x + f_{n-1}(x)y^{n-1}x \cdots + f_1(x)yx + f_0(x)x \\ &= q^{-n} f_n(x)xy^n + q^{-(n-1)} f_{n-1}(x)xy^{n-1} \cdots + q^{-1} f_1(x)xy + f_0(x)x \end{aligned}$$

The difference $q^{-n}xa - ax \in I$, so

$$(q^{-n} - q^{-(n-1)})f_{n-1}(x)xy^{n-1} \cdots + (q^{-n} - q^{-1})f_1(x)xy + (q^{-n} - 1)f_0(x)x \in I.$$

By induction hypothesis, I must contain a unit.

Lemma 2.4. *$(x\mathcal{O}_q(k^2))^i(y\mathcal{O}_a(k))^j = x^i y^j \mathcal{O}_q(k)$ for all $i, j \geq 0$.*

Proof. Let $x^i y^j(\sum f_\ell(x)y^\ell) \in x^i y^j \mathcal{O}_q(k^2)$. Using the relation in $\mathcal{O}_q(k^2)$, we get

$$x^i y^j(\sum f_\ell(x)y^\ell) = x^i(\sum f'_\ell(x)y^\ell)y^j = (\sum x^i f_\ell(x)y^\ell)y^j \in (x\mathcal{O}_q(k^2))^i(y\mathcal{O}_q(k^2))^j.$$

On the other hand, consider

$$(x\sum_{\ell_1} f_{\ell_1}(x)y^{\ell_1})\cdots(x\sum_{\ell_i} f_{\ell_i}(x)y^{\ell_i})(y\sum_{m_1} g_{m_1}(x)y^{m_1})\cdots(y\sum_{m_j} g_{m_j}(x)y^{m_j}) \in (x\mathcal{O}_q(k^2))^i(y\mathcal{O}_q(k^2))^j.$$

Hence

$$\begin{aligned} &(x\sum_{\ell_1} f_{\ell_1}(x)y^{\ell_1})\cdots(x\sum_{\ell_i} f_{\ell_i}(x)y^{\ell_i})(y\sum_{m_1} g_{m_1}(x)y^{m_1})\cdots(y\sum_{m_j} g_{m_j}(x)y^{m_j}) \\ &= (x^i\sum_{\ell_1} q^h f_{\ell_1}(x)y^{\ell_1}\cdots\sum_{\ell_i} f_{\ell_i}(x)y^{\ell_i})(y^j\sum_{m_1} g'_{m_1}(x)y^{m_1}\cdots\sum_{m_j} g'_{m_j}(x)y^{m_j}) \\ &= (x^i y^j\sum_{\ell_1} q^h f'_{\ell_1}(x)y^{\ell_1}\cdots\sum_{\ell_i} f'_{\ell_i}(x)y^{\ell_i}\sum_{m_1} g'_{m_1}(x)y^{m_1}\cdots\sum_{m_j} g'_{m_j}(x)y^{m_j}) \in x^i y^j \mathcal{O}_q(k^2). \end{aligned}$$

□

Let P be a prime ideal of $\mathcal{O}_q(k^2)$. Since $\mathcal{O}_q((k^x)^2) = \mathcal{O}_q(k^2)[x^{-1}, y^{-1}]$ is simple, P must contain a product $x^i y^j \in P$. This implies that $x^i y^j \mathcal{O}_q(k^2) \subseteq P$. By the last lemma, $(x\mathcal{O}_q(k^2))^i (y\mathcal{O}_q(k^2))^j \subseteq P$. Thus $x \in P$ or $y \in P$ because P is prime.

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