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ALGUNAS GENERALIZACIONES DE LA TEORÍA DE ANILLOS
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TO MY FAMILY
AND FRIENDS
This wouldn't have been possible without you.

Introducción

Dado un *anillo* asociativo con uno R podemos considerar su *categoía de módulos* R-Mod que consiste de todos los R-módulos izquierdos y de todos los R-morfismos entre ellos.La estructura multiplicativa de R lo hace tanto R-módulo izquierdo como derecho. Entonces cuando estudiamos la categoría R-Mod se espera obtener mucha información del anillo.

Muchas definiciones en anillos vienen de definiciones dadas en sus módulos, por ejemplo: Se dice que un anillo R es *semisimple* si lo es como R-módulo sobre si mismo. Pero hay definiciones en anillos que no están dadas en módulos.

Lo que se quiere hacer en este trabajo es tomar ciertas definiciones en anillos que no vienen de definiciones en módulos y trasladarlas al contexto de módulos, ésto para generalizar resultados en teoría de anillos. Entonces, si queremos tratar a un módulo como un anillo debemos cambiar un poco nuestro ambiente de trabajo.

En [31] se define y se describe *la categoría* $\sigma[M]$ o *catrgoría de Wisbauer* donde M es un R-módulo izquierdo. Esta categoría es la subcategoría plena de R-Mod que consiste de todos R-módulos M*-subgenerados.* La categoría $\sigma[M]$ es una categoría de Grothendieck, es decir, es completa, cocompleta, siempre existe un generador, existen cápsulas inyectivas, etcétera. Entonces, $\sigma[M]$ es una categoría que se comporta muy similar a R-Mod.

La categoría $\sigma[M]$ es más general que R-Mod en el sentido de que si ponemos $R = M$, entonces $\sigma[M] = R$-Mod. Si se leen cuidadosamente muchas definiciones de anillos nos podemos dar cuenta que éstas se pueden interpretar en términos de morfismos; esta observación es la que nos permite generalizar a módulos esas definiciones. Por ejemplo, también en [31] se pueden encontrar definiciones como V*-anillo, anillo regular, anillo coherente,* etcétera presentadas en términos de módulos.

Muchas otras definiciones en anillos están ligadas a la *estructura mul-*

tiplicatva del anillo. Una de éstas es, por ejemplo *nilpotencia*. Otra es el concepto de *primitud* el cual inspira mucho de este trabajo.

Trabajando con anillos asociativos con uno, no necesariamente conmutativos, tenemos ideales izquierdos, derechos y bilaterales. En este trabajo se entenderá por *ideal* un ideal bilateral y en otro caso se especificará de qué lado estamos tomando los ideales. Desde nuestros cursos de licenciatura estamos en contacto con el concepto de *elemento primo* en diferentes estructuras. Recordemos que un ideal P de un anillo R es un *ideal primo* si siempre que $IJ \leq P$ con I y J ideales de R entonces $I \leq P$ o $J \leq P$. Podemos ver que esta definición está basada en el producto del anillo R.

Muchos autores han tratado de definir *submódulos primos* para generalizar ideales primos, por ejemplo, [9], [30] , [3] and [22]. La definición con la que vamos a trabajar es la que fue dada en [22]. En ese artículo los autores definen un producto de un *submódulo totalmente invariante* de un módulo dado con cualquier R-módulo. Este producto es descrito usando un *prerradical* llamado α (ver [21]) que se define como sigue:

Dado un submódulo totalmente invariante $N \leq M$ y un R-módulo X

$$\alpha_N^M(X) = \sum\{f(N) | f : M \to X\}$$

Si N y L son submódulos totalmente invariantes de un módulo M su producto se define como

$$K_M L = \alpha_K^M(L)$$

También en [22] se muestra que si I y J son ideales de un anillo R entonces $IJ = I_R J$. Así, con este producto, ellos definen módulos y submódulos primos y semiprimos en la manera obvia.

Ha habido mucho trabajo desde que esta definición fue dada, hay artículos que generalizan resultados clásicos tales como *correspondencia local de Gabriel* [7], *dimension de Krull* para módulos [8] y módulos noetherianos *totalmente acotados* (FBN) [6].

En esta tesis queremos continuar en esta línea de investigación y principalmente tratar de encontrar el análogo de *anillos semiprimos de Goldie* izquierdos y del famoso *Teorema de Goldie* en el contexto de módulos.

El Teorema de Goldie dice que un anillo R tiene anillo clásico de cocientes izquierdo semisimple artiniano si y sólo si R es un anillo semiprimo con dimensión uniforme finita y satisface ACC en anuladores izquierdos. Wisbauer en [32, Teorema 11.6] prueba una versión del Teorema de Goldie en términos de módulos. Para un módulo retractable M con $S = End_R(M)$

las siguientes condiciones son equivalentes: *1.* M es no M-singular con dimensión uniforme finita y S es semiprimo, *2.* M es no M-singular con dimensión uniforme finita y para cada $N \leq_e M$ existe un monomorfismo $M \to N$, *3.* $End_R(\widehat{M})$ es semisimple artiniano y es el anillo clásico de cocientes izquierdo de S, aquí $\widehat{M}$ denota la cápsula M-inyectiva de M. Por otro lado, en [13], los autores estudian cuándo el anillo de endomorfismos de un módulo semiproyectivo es un anillo semiprimo de Goldie.

Para obtener una definición de *módulo de Goldie* que extienda la definición clásica de anillo de Goldie izquierdo, introducimos qué significa que un módulo satisfaga la *condicion de cadena ascendente en anuladores izquierdos*. Un *anulador izquierdo* en M es un submódulo de la forma $\mathcal{A}_X = \bigcap_{f \in X} Ker(f)$ para algún $X \subseteq End_R(M)$. Esta definición con $M = R$ es la definición usual de anulador izquierdo.

Entonces, un R-módulo M es un módulo de Goldie si M satisface ACC en anuladores izquierdos y tiene dimensión uniforme finita. En el Capítulo 4 se prueban unas caracterizaciones de módulos semiprimos de Goldie (Teorema 4.1.10, Teorema 4.1.25 y Corolario 4.1.26) que generalizan el Teorema de Goldie y extienden a los resultados presentados en [32, Teorema 11.6] y [13, Corolario 2.7].

Esta tesis está organizada en cuatro capítulos y un apéndice. El primer capítulo es el material necesario para la comprencíon de los demás capítulos. En el apéndice se dan resultados generales de una estructura ordenada llamada *casi-cuantal* que serán aplicados en las secciones 2.3 y 2.4.

El capítulo 2 está dedicado al estudio de submódulos primos y semiprimos. En las secciones 2.1 y 2.2 se presentan resultados generales de módulos semiprimos. En las siguientes dos secciones 2.3 y 2.4 se desarrolla una generalización de submódulos primos que da como consecuencia que el conjunto de submódulos semiprimos sea un *marco*.

En el capítulo 3, se dan varios resultados relacionados a módulos que satisfacen ACC en anuladores izquierdos. Este capítulo es importante porque muchos resultados referemtes a la estructura de los módulos semiprimos de Goldie estarán basados en este capítulo.

En el capítulo 4, se encuentran los principales resultados de esta tesis que generalizan el Teorema de Goldie. En la sección 4.2, se dan unas descomposiciones de la cápsula M-inyectiva de M cuando M un módulo semiprimo de Goldie. Estas descompocisiones están dadas en términos de los submódulos primos mínimos de M. También en esta sección se presentan algunos ejemplos de módulos semiprimos de Goldie.

Introduction

Given an associative *ring R with unitary element* we can consider its *Module category* R-Mod consisting of all left R-modules and the R-homomorphisms. The multiplicative structure on R makes it both left and right R-module, so when we study the category R-Mod, we expect to get much information of the ring through its modules.

Many definitions on rings come from definitions on its modules, for example: It is said a ring R is *semisimple* if it is semisimple as R-module over itself. But there are definitions on rings that they are not given on modules.

What we want to do in this work is to deal with some definitions given on rings but that do not come from modules and translate them to modules in order to generalize results on ring theory. So, if we want to treat a module as a ring we have to change our place of work a few.

In [31] is defined and described *the category $\sigma[M]$* or *Wisbauer's category* where M is a left R-module. This new category is the full subcategory of R-Mod consisting of all R-modules *M-subgenerated*. The category $\sigma[M]$ is a Grothendieck category so it is complete, co-complete, always exists a generator, exist injective hulls, etc. Then $\sigma[M]$ is a category that looks like R-Mod in some way.

The category $\sigma[M]$ is more general than R-Mod in the sense that if we put $R = M$ then $\sigma[M] = R$-Mod. If we read carefully many definitions on a ring we can realize that those definitions can be interpreted in terms of morphisms; this observation lets us to be able to generalize those definitions to modules. For example, also in [31] can be found definitions like: *V-rings, regular rings, coherent rings,....* etc. all in terms of modules.

Others definitions on rings are very related to the multiplicative structure of the ring. One of these is, for example *nil-potency*. Other is *primeness*, and is this condition what inspire much of this thesis.

When we work in an associative ring with unitary element, no necessary commutative, we have left, right and bilateral ideals. In this work we will mean *ideal* for a two-sided ideal and in other case the side will be written. Since undergraduated studies we are in contact with definitions of *prime elements* in some structures. Remember that an ideal P is a *prime ideal* if whenever $IJ \leq P$ with I, J ideals of R then $I \leq P$ or $J \leq P$. So, we can see that this definition is totally based on the product of the ring R.

Many authors have tried to define *prime submodules* in order to generalize prime ideals, for example: [9], [30] , [3] and [22]. The definition we will work with is that given in [22]. In this paper the authors define a product of *fully invariant submodules* of a given module M and any R-module. This product is described using a *preradical* called α (see [21]) that is defined as:

Given $N \leq M$ a fully invariant submodule and X an R-module

$$\alpha_N^M(X) = \sum\{f(N)|f : M \to X\}$$

Hence, if N and L are fully invariant submodules of M their product is defined as:

$$K_M L = \alpha_K^M(L)$$

Also in [22] is noticed that if I and J are ideals of a ring R then $IJ = I_R J$. Hence, with this product, they define prime and semiprime submodules and modules in the obvious way.

Much work have been done since this definition was given, there are papers which generalize classical results concern to primeness such as *local Gabriel correspondence* [7], *Krull dimension* for modules [8] and *fully bounded noetherian* (FBN) modules [6].

In this work, we want to keep this line and try to find the module theoretic analogous of *semiprime left Goldie* ring and the famous *Goldie's Theorem* principally.

Goldie's Theorem states that a ring R has a semisimple artinian classical left quotient ring if and only if R is a semiprime ring with finite uniform dimension and satisfies ACC on left annihilators. Wisbauer proves in [32, Theorem 11.6] a version of Goldie's Theorem in terms of modules. For a retractable R-module M with $S = End_R(M)$ the following conditions are equivalent: *1.* M is non M-singular with finite uniform dimension and S is semiprime, *2.* M is non M-singular with finite uniform dimension and for every $N \leq_e M$ there exists a monomorphism $M \to N$, *3.* $End_R(\widehat{M})$ is semisimple left artinian and it is the classical left quotient ring of S, here $\widehat{M}$

denotes the M-injective hull of M. Also, in [13] the authors study when the endomorphism ring of a semiprojective module is a semiprime Goldie ring.

In order to have a definition of *Goldie Module* such that it extends the classical definition of left Goldie ring, it is introduced what *ascending chain condition (ACC) on left annihilators* means on a module. A *left annihilator* in M is a submodule of the form $\mathcal{A}_X = \bigcap_{f\in X} Ker(f)$ for some $X \subseteq End_R(M)$. This definition with $R = M$ is the usual concept of left annihilator.

So, an R-module M is a Goldie module if M satisfies ACC on left annihilators and has finite uniform dimension. In chapter 4 are proved some characterizations of semiprime Goldie modules (Theorem 4.1.10, Theorem 4.1.25 and Corollary 4.1.26) which generalize the Goldie's Theorem and extend [32, Theorem 11.6] and [13, Corollary 2.7].

This thesis is organized in four chapters and one appendix. First chapter is the general background for the following chapters. In the appendix A is given general results in an ordered structure called *quasi-quantale* which will be applied in sections 2.3 and 2.4.

Chapter 2 is concern to study prime and semiprime submodules. In sections 2.1 and 2.2 are given general results about semiprime modules. In next two section 2.3 and 2.4 is developed a generalization of prime submodules that gives as consequence that the set of semiprime submodules is a *frame*.

In Chapter 3, are presented many results on modules which satisfy ACC on left annihilators. This chapter is very important because when we talk about the structure of semiprime Goldie modules many results will be supported on this chapter.

In Chapter 4, are presented the main theorems of this thesis that generalize Goldie's Theorem. In Section 4.2, it is presented some decompositions of the M-injective hull of M with M a semiprime Goldie module. These decompositions are given in terms of the minimal prime in M submodules. Also, in this section are presented some examples of semiprime Goldie modules.

Contents

Chapter 1

Preliminaries

As it was mentioned in the Introduction, in this chapter some background will be developed for the understanding of this thesis. Also it will presented the general notation that will be used.

By a *ring* R we will mean an associative ring with unitary element and all R-modules will be unitary. We will work with left R-modules and the morphism will be written from the left. Given two R-modules M and N, the set of homomorphism frome M to N is denoted by $Hom_R(M,N)$ and the endomorphism ring of a module M by $End_R(M)$

Let M be an R-module. If N is a submodule of M it will be denoted by $N \leq M$; if $m \in M$ the *cyclic submodule generated* by m is denoted by Rm. Recall that $N \leq M$ is *essential* if $N \cap L \neq 0$ for all $0 \neq L \leq M$, and I denote it by $N \leq_e M$. On the other hand, $N \leq M$ is *fully invariant* , denoted by $N \leq_{fi} M$ if for all $f \in End_R(M)$, $f(N) \leq N$. General knowledge of module theory is assumed and the reader is referred to [14], [1] and [28].

Given an R-module M, it is well known that

$$\Lambda(M) = \{N \leq M\}$$

is a complete lattice (see [28, Chapter III]) where the order is given by the inclusion, the suprema is the sum of submodules and the infima is the intersection. If we consider

$$\Lambda^{fi}(M) = \{N \in \Lambda(M) | N \leq_{fi} M\}$$

then it is a complete sub-lattice of $\Lambda(M)$.

As it was said in the Introduction, we will work on the category $\sigma[M]$. For this let us remember some definitions:

Definition 1.0.1. Let M, N be R-modules. It is said that N is M-generated if N is a homomorphic image of a direct sum of copies of M. In other words, there exist a set X and an epimorphism $M^{(X)} \to N$. Now, an R-module L is M*-subgenerated* if L can be embedded in an M-generated module.

If M is fixed and L is any R-module then L contains a largest M-generated submodule called the *trace* of M in L and is denoted by $tr^M(L)$. This submodule is defined as follows

$$tr^M(L) = \sum \{f(M) | f \in Hom_R(M, L)\}$$

Definition 1.0.2. Let M be an R-module. The category $\sigma[M]$ is the full subcategory of R-Mod consisting of all R-modules M-subgenerated.

Given M, the category $\sigma[M]$ is closed under submodules, direct sums and factor modules, in other words, it is a pre-torsion class. In [31] is proved that $\sigma[M]$ is a Grothendieck category, in particular $\sigma[M]$ has products and every module in $\sigma[M]$ has an injective hull. Since $\sigma[M]$ always have a generator, given $\{N_i\}_I$ a family in $\sigma[M]$ its product is described as

$$\prod_I^{[M]} N_i = tr^G(\prod_I N_i)$$

where $\prod_I N_i$ is the product in R-Mod and G is a generator of $\sigma[M]$. If $N \in \sigma[M]$ its injective hull in $\sigma[M]$ or its M-injective hull is described as

$$\widehat{N} = tr^M(E(N))$$

where $E(N)$ is the injective hull of N in R-Mod. For the proof of the universal properties of this objects and for a general background of $\sigma[M]$ see [31].

Other definition that has a principal roll in this work is the concept of *singularity*. Recall that an R-module K is *singular* if $ann_l(x) \leq_e R$ for all $x \in K$, here $ann_l(x)$ is the *left annihilator* of x in R. In the context of $\sigma[M]$ we have the following definition:

Definition 1.0.3. Let N be an R-module in $\sigma[M]$. It is said that N is M-singular if there exists a short exact sequence in $\sigma[M]$

$$0 \to K \to L \to N \to 0$$

such that $K \leq_e L$.

Now, we can see the next

Proposition 1.0.4. *Let N be an R-module. Then N is singular if and only if N is R-singular.*

Let $\mathcal{S}$ be the subclass of $\sigma[M]$ consisting of all M-singular modules in $\sigma[M]$. Given N in $\sigma[M]$, N has a largest M-singular submodule $\mathcal{Z}(N)$ defined as

$$\mathcal{Z}(N) = \sum\{tr^Z(N) | Z \in \mathcal{S}\}$$

If $\mathcal{Z}(N) = 0$ then we say that N is *non M-singular.*

Remark 1.0.5. Note that if $\sigma[X] \subseteq \sigma[Y]$ and N in $\sigma[X]$ is non Y-singular then N is non X-singular.

For more information about the class $\mathcal{S}$ and M-singular modules see [32].

Recall that a pair of non-empty classes $\tau = (\mathcal{T}, \mathcal{F})$ of R-modules is a *torsion theory* if it satisfies the following:

1. $Hom_R(T, F) = 0$ for all $T \in \mathcal{T}$ and for all $F \in \mathcal{F}$.
2. If $Hom_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$.
3. If $Hom_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$.

If in addition $\mathcal{T}$ is closed under submodules then we say that τ is a *hereditary torsion theory*.

If $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory, then $\mathcal{T}$ is closed under submodules, factor modules, direct sums, and extensions, and $\mathcal{F}$ is closed under submodules, products, injective hulls and extensions; $\mathcal{T}$ is called the torsion class and $\mathcal{F}$ the torsion-free class. The class of hereditary torsion theories in R-Mod is denoted by R-tors. In [11] it is proved that R-tors is a set, moreover it is a *frame*. Let us recall the definition of frame:

Definition 1.0.6. Let A be a complete lattice. It is said that A is a frame if the following distributive law holds:

$$\left(\bigvee X\right) \wedge a = \bigvee \{x \wedge a | x \in X\}$$

For all $X \subseteq A$ and all $a \in A$.

R-tors is a frame with the order given by the inclusion of torsion classes. The infima is given by the intersection. If $\mathcal{C}$ is any class of R-modules then there exist the least hereditary torsion theory such that $\mathcal{C}$ is contained in the torsion class and the greatest hereditary torsion theory where $\mathcal{C}$ is contained in the torsion-free class, these classes are denoted by $\xi(\mathcal{C})$ and $\chi(\mathcal{C})$ respectively. The hereditary torsion theory $\xi(\mathcal{C})$ is call the *hereditary torsion theory generated* by $\mathcal{C}$ and $\chi(\mathcal{C})$ is called the *hereditary torsion theory cogenerated* by $\mathcal{C}$. For more details in hereditary torsion theories in R-Mod see [11].

In [32] is proved that a hereditary torsion theory ρ in $\sigma[M]$ is just $\rho = \tau \cap \sigma[M]$ for some $\tau \in R$-tors. So, if M-tors denotes the set of hereditary torsion theories in $\sigma[M]$ then it is frame. The torsion theory generated and cogenerated by a class of modules in $\sigma[M]$ will be denoted as the case of R-tors.

Chapter 2

Primeness and Semiprimeness

2.1 Prime and Semiprime Modules

One of the study lines in general algebra is the study of prime objects, since the antiques Greeks until now. Our first encounter with primeness is, maybe, the prime numbers. These prime numbers take us to The Fundamental Theorem of Arithmetic showing themselves as the fundamental particles of arithmetic.

It can be defined prime element in others rings not just in $\mathbb{Z}$, and generalizing this idea we can take it to ideals of a commutative ring. So, let us recall the definitions of prime element and prime ideal in a commutative ring:

Definition 2.1.1. Let R be a commutative ring and $1 \neq p \in R$. It is said p is *prime* if whenever $p|ab$ with $a, b \in R$ then $p|a$ or $p|b$.

If we suppose R is a principal ideal domain (PID) then this definition is equivalent, in terms of ideals, to the following:

$p \in R$ is prime if whenever $ab \in Rp$ with $a, b \in R$ then $a \in Rp$ or $b \in Rp$.

So, we can generalize the definition of prime to prime ideal.

Definition 2.1.2. Let R be a commutative ring and P be a proper ideal of R. P is a *prime ideal* if whenever $ab \in P$ with $a, b \in R$ then $a \in P$ or $b \in P$. The set of prime ideals of R is called the *spectrum* of R and it is denoted by $Spec(R)$.

With this definitions we have

Proposition 2.1.3. *Let R be a commutative ring and P be an ideal of R.*

- *P is a prime ideal if and only if R/P is a integral domain.*
- *P is a maximal ideal if and only if R/P is a field.*
- *Every maximal ideal is a prime ideal.*

Last proposition ensures that every commutative ring contains a prime ideal.

For a commutative ring we have the next equivalence

Proposition 2.1.4. *Let P be an ideal in a commutative ring R. Then P is a prime ideal if and only if for every ideals $I, J \leq P$ such that $IJ \leq P$ implies $I \leq P$ or $J \leq P$.*

Proof. $\Rightarrow$. Let I, J be ideals of R such that $IJ \leq P$. Suppose that $I \nleq P$, so there exists $a \in I$ but $a \notin P$. Let $b \in J$ then $ab \in P$, since $a \notin P$ then $b \in P$. Thus $J \leq P$.

$\Leftarrow$. Let $a, b \in R$ such that $ab \in P$. Since R is commutative $RaRb \leq P$. Then, by hypothesis $Ra \leq P$ or $Rb \leq P$. Thus $a \in P$ or $b \in P$. □

For an arbitrary ring, the definition of prime ideal is a little different. Here is an example of why:

Example 2.1.5. Consider the ring $R = M_2(K)$ consisting of all 2×2 square matrices with coefficients in a field K. It is known that R is a simple ring, so 0 is the only maximal two-sided ideal.

In the commutative case, by Proposition 2.1.3 every maximal ideal is a prime ideal but here 0 does not satisfies the condition of definition 2.1.2 because

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

Let us give the definition of prime ideal in the general case:

Definition 2.1.6. Let R be a ring and P a proper ideal. P is a *prime ideal* if whenever $IJ \leq P$ for I, J ideals of R then $I \leq P$ or $J \leq P$. As in the commutative case, the the set of prime ideals of R will be called *spectrum* and denoted by $Spec(R)$.

Notice, by Proposition 2.1.4, that in the commutative case Definition 2.1.2 and Definition 2.1.6 are equivalent.

Remark 2.1.7. Let R be a ring. If $\mathcal{M}$ is a maximal ideal then $\mathcal{M}$ is a prime ideal. In fact, suppose that I and J are ideals which are not contained in $\mathcal{M}$, so $\mathcal{M}+I=R=\mathcal{M}+J$. Hence $R=(\mathcal{M}+I)(\mathcal{M}+J)=\mathcal{M}+IJ$. Thus IJ is not contained in $\mathcal{M}$.

Now, I would like to take the notion of prime ideal to submodules of an R-module. Before this, let us examine the product of ideals.

Recall that for every R-module M there exists an isomorphism

$$M \cong Hom_R(R,M)$$

So, every subset $X \subseteq R$ can be identified as a subset of $End_R(R)$. Then if $a,b \in R$ the product ab can be viewed as the evaluation $(\_ \cdot b)(a)$ where $(\_ \cdot b)$ is the morphism multiply by b from the right.

If I and J are left ideals of R the elements of IJ are finite sums of the form $\sum a_i b_i$ with $a_i \in I$ and $b_i \in J$, hence

$$\sum a_i b_i = \sum (\_ \cdot b_i)(a_i)$$

where $(\_ \cdot b_i) : R \to J$ is the R-morphism multiplication by b_i. Then

$$IJ = \sum \{f(I) | f : R \to J\}$$

Other thing that we have to consider is that: in Definition 2.1.6 I and J are ideals, i.e. two-sided ideals, if we want to translate the notion of prime ideal we need to work with the analogous, in a module, of two-sided ideal.

Definition 2.1.8. Let M be an R-module and $N \leq M$. The submodule N is *fully invariant* in M if $f(N) \leq N$ for all $f \in End_R(M)$.

Then, a left ideal of R is fully invariant if and only if it is a two-sided ideal.

With all this in mind, we can define a product of submodules of a given module. Next definition appeared first in [4].

Definition 2.1.9. Let M be an R-module and K, L submodules of M. The product of K with L in M is defined as

$$K_M L = \sum \{f(K) | f \in Hom_R(M,L)\}$$

Remark 2.1.10. Notice that this product can be defined for $K \leq M$ and an R-module X as follows:

$$K_M X = \sum\{f(K)|f : M \to X\}$$

Recall the definition of a preradical on a ring R:

Definition 2.1.11. A *preradical* ρ on a ring R is a subfunctor of the identity functor on R-Mod, i.e., for all R-module M, $\rho(M) \leq M$ and for all morphism $f : M \to N$, $\rho(f) = f|_{\rho(M)} : \rho(M) \to \rho(N)$.

Remark 2.1.12. If $K \leq M$ and X is an R-module, then $K_{M\_}$ is a preradical. Following [21], if $K \leq_{fi} M$ and X is an R-module the preradical *alpha* is defined as:

$$\alpha_K^M(X) = \sum\{f(K)|f : M \to X\}$$

So, when $K \leq_{fi} M$, $K_{M\_} = \alpha_K^M(\_)$.

In [22], given two fully invariant submodules K, L of M their product in M is defined as $KL := \alpha_K^M(L)$.

This product satisfies the following properties:

Lemma 2.1.13. *Let M be an R-module and $K, K' \leq M$. Then:*

1. *If $K \subseteq K'$ then $K_M X \subseteq K'_M X$ for every module X.*
2. *If X is a left module and $Y \subseteq X$ then $K_M Y \subseteq K_M X$.*
3. *$K_M X \subseteq X$ for every module X.*
4. *$K_M X = 0$ if and only if $f(K) = 0$ for all $f \in Hom_R(M, X)$.*
5. *$0_M X = 0$, for every module X.*
6. *Let $\{X_i \mid i \in I\}$ be a family of submodules of M then $\sum_{i\in I} (K_M X_i) \subseteq K_M \left(\sum_{i\in I} X_i\right)$.*
7. *$\left(\sum_{i\in I} K_i\right)_M N = \sum_{i\in I} K_{iM} N$ for every family of submodules $\{K_i \mid i \in I\}$ of M.*
8. *If $\{X_i|i \in I\}$ is a non empty family of R-modules then $K_M(\bigoplus_{i\in I} X_i) = \bigoplus_{i\in I}(K_M X_i)$.*

Proof. For *1,2,3,4,5* and *8* see [7, Proposition 1.3].

6. Let $\{X_i|i \in I\}$ be a family of submodules of M. Since $X_i \leq \sum_{i\in I} X_i$, by (*2*) of this proposition $(K_M X_i) \subseteq K_M \left(\sum_{i\in I} X_i\right)$. Thus

$$\sum_{i\in I}(K_M X_i) \subseteq K_M \left(\sum_{i\in I} X_i\right)$$

7. Let $\{K_i|i \in I\}$ be a family the submodules of M. Then

$$\left(\sum_{i\in I} K_i\right)_M N = \sum \left\{ \left(\sum_{i\in I} K_i\right) |f \in Hom_R(M,N)\right\}$$

$$= \sum \left\{ \sum_{i\in I} f\left(K_i\right)|f \in Hom_R(M,N)\right\} = \sum_{i\in I} \sum \{f(K_i)|f \in Hom_R(M,N)\}$$

$$= \sum_{i\in I}(K_{iM} N)$$

□

Example 2.1.14 (The equality in 2.1.13.(*6*) is not true in general.)**.** Let $R = \mathbb{Z}$ and p a prime number. If $M = \mathbb{Q}$, $K = \mathbb{Q}$, $X = \mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q}|p \nmid b\}$ and $Y = \{\frac{a}{p^n} \in \mathbb{Q}|n \in \mathbb{N}\}$, then $X + Y = \mathbb{Q}$, $K_M X = \mathbb{Q}_M \mathbb{Z}_{(p)} = 0$ and $K_M Y = \mathbb{Q}_M Y = 0$. On the other hand, $K_M(X+Y) = \mathbb{Q}_M\mathbb{Q} = \mathbb{Q}$. Thus $K_M(X+Y) \nsubseteq (K_M X) + (K_M Y)$. Notice that $\mathbb{Q}$ is not projective in $\sigma[\mathbb{Q}] = \mathbb{Z} - Mod$.

At this point, I want to recall some lattice-theoretical definitions.

Definition 2.1.15. Let Q be a complete lattice, Q is a *quantale* if Q has a binary associative operation $\cdot : Q \times Q \to Q$ such that

$$l\left(\bigvee X\right) r = \bigvee \{lxr \mid x \in X\}$$

holds for all $l, r \in Q$ and $X \subseteq Q$.

Definition 2.1.16. A lattice A is *meet-continuous* if

$$a \wedge (\bigvee X) = \bigvee \{a \wedge x \mid x \in X\}$$

for all $a \in A$ and $X \subseteq A$ any directed set.

A is *modular* if $(a \vee c) \wedge b = a \vee (c \wedge b)$, for all $a, b, c \in A$ such that $a \leq b$.

A is an *idiom* if A is a meet-continuous and modular lattice.

For general theory of these lattices see [18], [28] and [26].

Remark 2.1.17. Notice that if $I, J \leq R$ then $IJ = I_R J$. We know, that the product of (left) ideals is associative and if $\{I_i\}_\Gamma$ is a non empty family of left ideals and $J \leq R$ then

$$(\sum_\Gamma I_i)J = \sum_\Gamma (I_i J)$$

and

$$J(\sum_\Gamma I_i) = \sum_\Gamma (J I_i)$$

That is, the lattice of left ideals of R, $\Lambda(R)$ is a quantale.

For an R-module M, the lattice $\Lambda(M)$ is an idiom (see [28, Chapter III]). By Example 2.1.14, $\Lambda(M)$ is not a quantale in general. In fact, next example shows $\_{M}\_$ is not always associative. To make this product associative we need and extra hypothesis: M is projective in $\sigma[M]$.

Example 2.1.18. Consider $\mathbb{Z}$ the additive group of integers and $\mathbb{Q}$ the additive group of rational numbers, then $0 = (\mathbb{Z}\,{}_{\mathbb{Q}}\,\mathbb{Z})_{\mathbb{Q}}\,\mathbb{Q} \neq \mathbb{Z}\,{}_{\mathbb{Q}}\,(\mathbb{Z}\,{}_{\mathbb{Q}}\,\mathbb{Q}) = \mathbb{Q}$.

Lemma 2.1.19. *Suppose M is projective in $\sigma[M]$. Then*

1. *The product $-_M- : \Lambda(M) \times \Lambda(M) \to \Lambda(M)$ is associative.*
2. *$K_M\left(\sum_{i\in I} X_i\right) = \sum_i (K_M X_i)$ for every directed family of submodules $\{X_i \mid i \in I\}$ of M.*

Proof. *1.* See [3, Proposition 5.6].

2. Let $\{X_i | i \in I\}$ be a directed family of submodules of M. Let $\sum_{j\in J} f_j(k_j) \in K_M\left(\sum_{i\in I} X_i\right)$, with $f_j : M \to \sum_{i\in I} X_i$. Since M is projective in $\sigma[M]$ for each f_j there exist $g_{i_j} : M \to X_i$ such that $\sum_{i\in I} g_{i_j}(k_j) = f_j(k_j)$, that is, the following diagram commutes

$$\begin{tikzcd}
& M \arrow[dl, "\bigoplus g_{i_j}"'] \arrow[d, "f_j"] \\
\bigoplus_{i\in I} X_i \arrow[r, two heads] & \sum_{i\in I} X_i
\end{tikzcd}$$

Then

$$\sum_{j\in J} f_j(k_j) = \sum_{j\in J}\sum_{i\in I} g_{i_j}(k_j) \in \sum X_{i_j}$$

Since this sum is finite and $\{X_i | i \in I\}$ is directed, there exists $l \in I$ such that $\sum X_{i_j} \subseteq X_l$. Thus

$$\sum_{j\in J} f_j(k_j) = \sum_{j\in J}\sum_{i\in I} g_{i_j}(k_j) \in K_M X_l \subseteq \sum_{i\in I}(K_M X_i)$$

The other contention follows from 2.1.13.(6). $\square$

Definition 2.1.20. Let A be a complete lattice. A is a *quasi-quantale* if A has an associative product $A \times A \to A$ such that for all directed subsets $X, Y \subseteq A$ and $a \in A$:

$$(\bigvee X)a = \bigvee\{xa \mid x \in X\}$$

$$a(\bigvee Y) = \bigvee\{ay \mid y \in Y\}$$

We say that A is a *left (resp. right, resp. bilateral) quasi-quantale* if there exists $e \in A$ such that $e(a) = a$ (resp. $(a)e = a$, resp. $e(a) = a = (a)e$) for all $a \in A$.

This definition generalize both quantales and idioms. Note that by Lemma 2.1.19 $\Lambda(M)$ is a quasi-quantale when M is projective in $\sigma[M]$. Quasi-quantales are treated in Apendix A.

Next definition extends Definition 2.1.6, it is one of the fundamental concepts in this work and was given first in [22].

Definition 2.1.21. Let M be an R-module. A proper fully invariant submodule $N < M$ is a *prime submodule* in M if for any fully invariant submodules $K, L \leq M$ such that $K_M L \leq N$, then $K \leq N$ or $L \leq N$. We say that M is a *prime module* if 0 is a prime submodule.

Notice that if when $M = R$ then the prime ideals are the prime submodules in R.

Remark 2.1.22. The set of prime submodules in M will be called the *spectrum of* M and denoted by $Spec(\Lambda^{fi}(M))$. The notation is because in section 2.3 we will mention other spectra associated to M.

Next proposition characterizes prime submodules of a module M projective in $\sigma[M]$.

Proposition 2.1.23. *Let M be projective in $\sigma[M]$ and P a fully invariant submodule of M. The following conditions are equivalent:*

1. *P is prime in M.*

2. *For any submodules K, L of M such that $K_M L \leq P$, then $K \leq P$ or $L \leq P$.*

3. *For any submodules K, L of M containing P and such that $K_M L \leq P$, then $K = P$ or $L = P$.*

4. *M/P is a prime module.*

Proof. *1* $\Rightarrow$ *2*. It follows from [7, 1.11].

2 $\Rightarrow$ *3*. It is clear.

3 $\Rightarrow$ *1*. Suppose that K, L are submodules of M such that $K_M L \leq P$.

We claim that $K_M(L+P) \leq P$. Since $K_M L \leq L \cap P$, by [3, Proposition 5.5] $K_M(L/L \cap P) = 0$ so $K_M(L + P/P) = 0$. Thus $K_M(L+P) \leq P$.

On the other hand,

$$(K+P)_M(L+P) = K_M(L+P) + P_M(L+P) \leq P$$

because P is fully invariant in M.

Then, by hypothesis $K+P = P$ or $L+P = P$, hence $K \leq P$ or $L \leq P$.

1 $\Leftrightarrow$ *4*. It follows from [22, Proposition 18]. □

Proposition 2.1.24. *Let M be an R-module. If M has at least one prime submodule then M has at least one minimal prime submodule.*

Proof. Let $P \leq M$ be a prime submodule. Consider $\Gamma = \{Q \leq P | Q \textit{ is prime}\}$. This family is not empty because $P \in \Gamma$. Let $\mathcal{C} = \{Q_i\}$ be a descending chain in Γ. Let $N, K \leq M$ be fully invariant submodules of M such that $N_M K \leq \bigcap \mathcal{C}$. Suppose that $N \nleq \bigcap \mathcal{C}$. Then there exists Q_j such that $N \nleq Q_j$ and $N \nleq Q_l$ for all $Q_l \leq Q_j$. Therefore $K \leq Q_l$ for all $Q_l \leq Q_j$, and since $\mathcal{C}$ is a chain then $K \leq \bigcap \mathcal{C}$. Therefore $\bigcap \mathcal{C} \in \Gamma$. By Zorn's Lemma Γ has minimal elements. □

Proposition 2.1.3.(*3*) says that every maximal ideal is a prime ideal. In general, for an R-module it is not true that a maximal fully invariant submodule is prime. See [7, Example 1.12, Proposition 1.13].

If R is a commutative ring, then $P < R$ is a prime ideal if and only if $R - P$ is a multiplicative set. When R is not commutative $R - P$ is not necessarily multiplicative but P is a prime ideal if and only if the complement of P, $R - P$ is an m-system [17, Corollary 10.4]. Recall that an *m-system* is a nonempty subset B of a ring R such that for all $b, b' \in B$ there exists $r \in R$ such that $brb' \in B$.

The last definition can be translate to the context of modules.

Definition 2.1.25. Let M be an R-module. A subset $B \subseteq M$ is an *M-m-system* if for all $b, b' \in B$ there exists a morphism $f : M \to Rb'$ such that $f(b) \in B$.

Remark 2.1.26. Note that if R is a ring and $B \subseteq R$, then B is an m-system if and only if B is an R-m-system.

Proposition 2.1.27. *Let M be projective in $\sigma[M]$. Let $P \leq M$ be a fully invariant submodule of M. Then P is prime in M if and only if $M - P$ is an M-m-system.*

Proof. $\Rightarrow$. Suppose that P is prime in M and let $a, b \in M - P$. Since $a, b \notin P$, by Proposition 2.1.23 $Ra_M Rb \nsubseteq P$, so there exists $f : M \to Rb$ such that $f(a) \notin P$. Thus $M - P$ is a M-m-system.

$\Leftarrow$. Let $a, b \in M$. Suppose that $Ra_M Rb \leq P$.If $Ra \nleq P$ and $Rb \nleq P$ then there exists $r, t \in R$ such that $rb, ta \in M - P$. Since $M - P$ is an M-m-system there exists $f : M \to R(rb)$ such that $f(ta) \in S$. Hence $R(ta)_M R(rb) \nleq P$ but $R(ta)_M R(rb) \leq Ra_M Rb \leq P$. Contradiction. $\square$

Proposition 2.1.28. *Suppose M is projective in $\sigma[M]$. Let $B \subset M$ be an M-m-system. If P is a fully invariant submodule of M maximal with the property $P \cap B = \emptyset$, then P is prime in M.*

Proof. Let K and L be fully invariant submodules of M with $P \leq K$ and $P \leq L$ such that $K_M L \leq P$. Suppose that $P \neq K$ and $L \neq P$. By maximality of P there exist $a \in K$ and $b \in L$ such that $a, b \in B$. Then there exists $f : M \to Rb$ with $f(a) \in B$, so $Ra_M Rb \nleq P$. Contradiction. By Proposition 2.1.23.(*3*) P is prime in M. $\square$

If R is a commutative ring and I is an ideal of R the *radical of* I, $\sqrt{I}$ is defined as

$$\sqrt{I} = \{r \in R | r^n \in I \textit{ for some } n \in \mathbb{N}\}$$

And it can be proved that $\sqrt{I} = \bigcap\{P \in Spec(R) | I \leq P\}$

In the non-commutative case the definition of the radical of an ideal is a little different.

Definition 2.1.29. Let I be an ideal of a ring R. The *radical of* I is defined as

$$\sqrt{I} = \{r \in R | \text{ every } m - system \text{ containing } r \text{ meets } I\}$$

It is easy to see that $\sqrt{I} \subseteq \{r \in R | r^n \in I \text{ for some } n \in \mathbb{N}\}$. What is not clear to see is if $\sqrt{I}$ is an ideal or not but we have next Theorem [17, Theorem 10.7]

Theorem 2.1.30. *For any ring R and any ideal $I \leq R$, $\sqrt{I}$ equals the intersection of all the prime ideals containing I. In particular $\sqrt{I}$ is an ideal of R.*

Now, we can put the definition and theorem analogous to Definition 2.1.29 and Theorem 2.1.30 on modules

Definition 2.1.31. Let N be a fully invariant submodule of M. The radical of N in M is

$$\sqrt{N} = \{b \in M | \text{Every } M - m - system \text{ containing } b \text{ intersects } N\}$$

Proposition 2.1.32. *Let M be projective in $\sigma[M]$. Let N a fully invariant submodule of M, then*

$$\sqrt{N} = \bigcap\{P \in Spec(\Lambda^{fi}(M)) | N \subseteq P\}$$

Proof. Suppose that $b \notin \sqrt{N}$, so there exists an M-m-system B such that $b \in B$ and $N \cap B = \emptyset$. By Zorn's Lemma there exists $P \leq M$ fully invariant such that $N \leq P$ and P is maximal with respect to $P \cap B = \emptyset$. By Proposition 2.1.28 P is prime in M. Hence $b \notin \bigcap\{P \in Spec(\Lambda^{fi}(M)) | N \subseteq P\}$.

Now, if $b \notin \bigcap\{P \in Spec(\Lambda^{fi}(M)) | N \subseteq P\}$, there exists P prime in M containing N such that $b \notin P$. So $b \in M - P$ and $M - P$ is a m-system by Proposition 2.1.27. Since $N \leq P$ then $N \cap (M - P) = \emptyset$, so $s \notin \sqrt{N}$. □

Definition 2.1.33. Let M be an R-module. The *lowest radical* of M is defined as $Nil_*(M) = \sqrt{0}$.

Definition 2.1.34. Let M be an R-module. A proper fully invariant submodule $N < M$ is a *semiprime submodule* in M if for any fully invariant submodule $K \leq M$ such that $K_M K \leq N$, then $K \leq N$. We say that M is a *semiprime module* if 0 is a semiprime submodule.

Example 2.1.35. Let N be a fully invariant submodule of M. Then $\sqrt{N}$ is a semiprime submodule in M by Proposition 2.1.32.

We know that an ideal I is semiprime if and only if $I = \sqrt{I}$. See [17, Theorem 10.11]. Also in modules there is an analogous theorem

Proposition 2.1.36. *Let M be projective in $\sigma[M]$ and N a fully invariant submodule of M. The following conditions are equivalent:*

1. *N is semiprime in M.*
2. *For any submodule K of M, $K_M K \leq N$ implies $K \leq N$.*
3. *For any submodule $K \leq M$ containing N such that $K_M K \leq N$, then $K = N$.*
4. *M/N is a semiprime module.*
5. *If $m \in M$ is such that $Rm_M Rm \leq N$, then $m \in N$.*
6. *$N = \sqrt{N}$.*

Proof. *1 ⇒ 2.* Let $K \leq M$ such that $K_M K \leq N$. Consider the submodule $K_M M$ of M. This is the minimal fully invariant submodule of M which contains K and $K_M X = (K_M M)_M X$ for every module X. Hence by Lemma 2.1.13

$$K_M K = (K_M M)_M K \leq ((K_M M)_M K)_M M) \leq N_M M$$

Since N is a fully invariant submodule of M then $N_M M = N$ and by [3, Proposition 5.5] $(K_M M)_M (K_M M) = ((K_M M)_M K)_M M) \leq N$. Since N is semiprime in M, $K_M M \leq N$. Hence $K \leq N$.

2 ⇒ 3. It is clear.

3 ⇒ 1. The proof is analogous to the proof of Proposition 2.1.23.

1 ⇔ 4. By [23].

2 ⇒ 5. By hypothesis.

5 ⇒ 6. Since N is proper in M, let $m_0 \in M \setminus N$. Then $Rm_{0M} Rm_0 \nleq N$. Now, let $0 \neq m_1 \in Rm_{0M} Rm_0$ but $m_1 \notin N$ Then $Rm_{1M} Rm_1 \nleq N$ and

$Rm_{1M}Rm_1 \leq Rm_{0M}Rm_0$. We obtain a sequence of non-zero elements of M, $\{m_0, m_1, ...\}$ such that $m_i \notin N$ for all i and $Rm_{i+1M}Rm_{i+1} \leq Rm_{iM}Rm_i$.

By Zorn's Lemma there exists a fully invariant submodule P of M with $N \leq P$, maximal with the property that $m_i \notin P$ for all i .

We claim P is a prime submodule. Indeed, let K and L submodules of M containing P properly. Since $P < K$ and $P < L$, then there exists m_i and m_j such that $m_i \in K$ and $m_j \in L$. Suppose $i \leq j$, then $Rm_{iM}Rm_i \leq K$ and by construction $m_j \in Rm_{iM}Rm_i$ and thus $m_j \in K$. If we put $k = max\{i, j\}$, then $m_k \in K$ and $m_k \in L$. Hence, $Rm_{kM}Rm_k \leq K_M L$, and so $K_M L \nleq P$. By Proposition 2.1.23.(*3*), P is prime in M.

6 $\Rightarrow$ *1*. It is clear. □

Remark 2.1.37. Note that by the proof of Proposition 2.1.36 if M is projective in $\sigma[M]$ and semiprime then M has prime submodules, so by Proposition 2.1.24 M has minimal prime submodules.

Minimal prime submodules will be crucial for developing Chapters 3 and 4.

Corollary 2.1.38. *Let $0 \neq M$ be a semiprime module and projective in $\sigma[M]$. Then*

$$0 = \bigcap\{P \leq M | P \textit{ is a minimal prime in } M\}$$

Proof. Let $x \in \bigcap\{P \leq M | P$ is a minimal prime in $M\}$ and $Q \leq M$ be a prime submodule in M. By Proposition 2.1.24 there exists a minimal prime submodule P such that $P \leq Q$ then $x \in Q$ and x is in the intersection of all primes in M. By Proposition 2.1.36, $x = 0$. □

As in the case of prime submodules, semiprime submodules can be characterized by their complement.

Recall that a subset A of a ring R is an *n-system* if for any $a \in A$ there exists $r \in R$ such that $ara \in A$. Then an ideal $I \leq R$ is semiprime if and only if $R - P$ is an n-system. See [17, pag. 157]

Definition 2.1.39. Let M be an R-module. A subset $A \subseteq M$ is an *M-n-system* if for any $a \in A$ there exists a morphism $f : M \to Ra$ such that $f(a) \in A$.

Proposition 2.1.40. *Suppose that M is projective in $\sigma[M]$. Let $N \leq M$ a fully invariant submodule of M. Then N is semiprime in M if and only if $M - N$ is an M-n-system.*

Proof. The proof is analogous to proof that Proposition 2.1.27. □

We can find (semi)prime submodules in terms of their morphisms, as the following proposition shows.

Proposition 2.1.41. *Let $S := End_R(M)$ and assume that M generates all its submodules. If N is a fully invariant submodule of M such that $Hom_R(M, N)$ is a prime (semiprime) ideal of S, then N is prime (semiprime) in M.*

Proof. Let K and L be fully invariant submodules of M such that $K_M L \leq N$. Put $I = Hom_R(M, L)$ and $J = Hom_R(M, K)$. Let $m \in M$ and $\sum f_i g_i \in IJ$. Since $g_i \in J$ and $g_i(m) \in K$ then $\sum f_i(g_i(m)) \in K_M L \leq N$. Hence $IJ \leq Hom_R(M, N)$. Since $Hom_R(M, N)$ is prime (semiprime) in S, then $I \leq Hom_R(M, N)$ or $J \leq Hom_R(M, N)$. Hence $tr^M(L) := Hom(M, L)M \leq N$ or $tr^M(K) \leq N$ and since M generates all its submodules then $tr^M(L) = L \leq N$ or $tr^M(K) = K \leq N$. Thus N is a prime (semiprime) submodule. □

Following [15] we give the next:

Definition 2.1.42. A module M is *retractable* if $Hom_R(M, N) \neq 0$ for all $0 \neq N \leq M$.

Notice that if M generates all its submodules then M is retractable.

Corollary 2.1.43. *Let $S := End_R(M)$ with M retractable. If S is a prime (semiprime) ring then M is prime (semiprime).*

Proof. Let K and L be fully invariant submodules of M such that $K_M L = 0$. Since $Hom_R(M, 0)$ is a prime (semiprime) ideal of S then by the proof of Proposition 2.1.41, $tr^M(K) = 0$ or $tr^M(L) = 0$. Since M is retractable, $K = 0$ or $L = 0$. Hence 0 is prime (semiprime) in M. Thus M is prime (semiprime). □

Lemma 2.1.44. *Let M be projective in $\sigma[M]$. If M is semiprime then M is retractable.*

Proof. Let $N \leq M$ and suppose $Hom_R(M, N) = 0$. So $M_M N = 0$ but $N_M N \subseteq M_M N = 0$. Since M is semiprime then $N = 0$. □

To end this section, I shall show a characterization of semiprime artinian modules.

Lemma 2.1.45. *Let M be an R-module and N a minimal submodule of M. Then $N_M N = 0$ or N is a direct summand of M.*

Proof. Suppose that $N_M N \neq 0$. Then there exists $f : M \to N$ such that $f(N) \neq 0$. Since $0 \neq f(M) \leq N$ and N is a minimal submodule, $f(M) = N$. On the other hand, $Ker(f) \cap N \leq N$, since $f(N) \neq 0$ then $Ker(f) \cap N = 0$. We have that $M/Ker(f) \cong N$ and since N is a minimal submodule, then $Ker(f)$ is a maximal submodule of M. Thus $Ker(f) \oplus N = M$. □

Corollary 2.1.46. *Let M be a retractable module. If N is a minimal submodule in a semiprime module M, then N is a direct summand.*

Proof. Since M is semiprime, $N_M N \neq 0$. □

Theorem 2.1.47. *The following conditions are equivalent for a retractable R-module M:*

1. *M is semisimple and left artinian.*
2. *M is semiprime and left artinian.*
3. *M is semiprime and satisfies the descending chain condition, DCC, on cyclic submodules and direct summands.*

Proof. *1* $\Rightarrow$ *2* : If M is semisimple, then it is semiprime.

2 $\Rightarrow$ *3* : Since M is left artinian, then it satisfies DCC on cyclic submodules and direct summands.

3 $\Rightarrow$ *1* : Since M satisfies DCC on cyclic submodules, there exists a minimal submodule K_1 of M. By Corollary 2.1.46, $M = K_1 \oplus L_1$. Now there exists a minimal submodule K_2 of L_1 such that $L_1 = K_2 \oplus L_2$. With this process we obtain a descending chain of direct summands, which is finite by hypothesis, say $L_1 \supseteq L_2 \supseteq L_3 \supseteq ... \supseteq L_m$. Since L_m is simple and $M = K_1 \oplus K_2 \oplus ... \oplus K_m \oplus L_m$, then M is semisimple.

Now, if M is semisimple and satisfies DCC on direct summands then M is artinian. □

2.2 Annihilator Submodules

In this section some particular submodules called *annihilator submodules* will be studied. Here we will focus on annihilator submodules of a semiprime module. So let us start with the definition of *annihilator* in our context.

Definition 2.2.1. Let $K \in \sigma[M]$. The *annihilator of K in M* is the submodule of M defined as

$$Ann_M(K) = \bigcap\{Ker(f)|f \in Hom_R(M,K)\}$$

This submodule was first defined in [3]. It can be shown that $Ann_M(K)$ is the greatest submodule of M such that $Ann_M(K)_M K = 0$.

When $N \leq M$ it is possible to define a right annihilator in M, in other words the greatest submodule $Ann^r_M(N)$ of M such that $N_M Ann^r_M(N) = 0$.

Definition 2.2.2. Let M be an R-module and $N \leq M$. The *right annihilator of N in M* is defined as

$$Ann^r_M(N) = \sum\{L \leq M|N_M L = 0\}$$

Proposition 2.2.3. *Let M be projective in $\sigma[M]$ and $N \leq M$. $Ann^r_M(N)$ is a fully invariant submodule of M and is the greatest submodule of M such that $N_M Ann^r_M(N) = 0$.*

Proof. Let $\{L_i\}_I$ be the family of submodules of M such that $N_M L_i = 0$. Note that if $N_M L_i = 0$, then $N_M(L_{iM}M) = 0$ because the product is associative. Hence, without lost of generality we can assume that every L_i is fully invariant. Then $Ann^r_M(N)$ is a fully invariant submodule of M.

Now, by Lemma 2.1.13.(*8*) $0 = \bigoplus_I(N_M L_i) = N_M(\bigoplus_I L_i)$. Since M is projective in $\sigma[M]$, $N_M(\frac{\bigoplus_I L_i}{A}) = 0$ for every $A \leq \bigoplus_I L_i$ by [3, Proposition 5.5]. Thus $N_M Ann^r_M(N) = 0$. □

The following lemma shows that, when M is a semiprime module projective in $\sigma[M]$ and $N \leq M$, there is no distinction between $Ann_M(N)$ and $Ann^r_M(N)$.

Lemma 2.2.4. *Let M be semiprime and projective in $\sigma[M]$. Let $N, L \leq M$. If $L_M N = 0$, then $N_M L = 0$ and $L \cap N = 0$.*

Proof. Since $L_M N = 0$, then

$$0 = N_M(L_M N)_M L = (N_M L)_M(N_M L).$$

Hence $N_M L = 0$.

Now, since $L \cap N \leq L$ and $L \cap N \leq N$, then

$$(L \cap N)_M(L \cap N) \leq L_M N = 0.$$

Thus $L \cap N = 0$ □

Corollary 2.2.5. *Let M be semiprime and projective in $\sigma[M]$. If $N \leq M$, then $N_M Ann_M(N) = 0$. This implies that $Ann_M(N) = Ann^r_M(N)$.*

Now we can define the submodules we are interested in.

Definition 2.2.6. Let M be an R-module and $N < M$. We say N is an *annihilator submodule* if $N = Ann_M(K)$ for some $0 \neq K \leq M$.

Proposition 2.2.7. *Let M be semiprime and projective in $\sigma[M]$ and $N \leq M$. Then N is an annihilator submodule if and only if $N = Ann_M(Ann_M(N))$.*

Proof. $\Rightarrow$: By Lemma 2.2.4 $N \leq Ann_M(Ann_M(N))$.

There exists $K \leq M$ such that $N = Ann_M(K)$, hence

$$K_M N = K_M Ann_M(K) = 0$$

and thus $K \leq Ann_M(N)$. Therefore,

$$Ann_M(Ann_M(N)) \leq Ann_M(K) = N$$

It follows that $N = Ann_M(Ann_M(N))$. □

Proposition 2.2.8. *Let M be projective in $\sigma[M]$ and semiprime. If $N \leq M$ then $Ann_M(N)$ is the greatest fully invariant submodule of M such that $Ann_M(N) \cap N = 0$. Moreover, $N \oplus Ann_M(N)$ intersects all fully invariant submodules of M.*

Proof. Let $L \leq M$ be a fully invariant pseudocomplement of N in M. Then

$$L_M N \leq L \cap N = 0$$

Thus $L \leq Ann_M(N)$. By Lemma 2.2.4 $Ann_M(N) \cap N = 0$. Thus $L = Ann_M(N)$. □

By Remark 2.1.37 if M is projective in $\sigma[M]$ and semiprime then M has minimal primes. In this case, the annihilator in M can be described in terms of minimal primes.

Proposition 2.2.9. *Let M be projective in $\sigma[M]$ and a semiprime module. Let $N \leq M$ and J be the set of all minimal prime submodules of M which does not contain N. Then $Ann_M(N) = \bigcap\{P | P \in J\}$.*

Proof. Put $K = \bigcap\{P | P \in J\}$. Any element in $K \cap N$ is in the intersection of all minimal prime submodules of M which is zero by Corollary 2.1.38. Then $K \cap N = 0$. Since K is fully invariant in M, $K \leq Ann_M(N)$ by Proposition 2.2.8. Now, let $P \in J$. Since $Ann_M(N)_M N = 0 \leq P$ and $N \nleq P$, then $Ann_M(N) \leq K$. □

Next proposition says that some annihilator submodules are minimal primes.

Proposition 2.2.10. *Let M be projective in $\sigma[M]$ and semiprime. The following conditions are equivalent for $N \leq M$:*

1. *N is a maximal annihilator submodule.*
2. *N is an annihilator submodule and is a minimal prime submodule.*
3. *N is prime in M and N is an annihilator submodule.*

Proof. *1* $\Rightarrow$ *2* : Let $K \leq M$ such that $N = Ann_M(K)$. Let $L, H \leq M$ be fully invariant submodules of M such that $L_M H \leq N$. Assume $H \nleq N$. Then $0 \neq H_M K$. Hence $Ann_M(K) \leq Ann_M(H_M K)$, but since $Ann_M(K)$ is a maximal annihilator submodule, then $Ann_M(K) = Ann_M(H_M K)$.

Since M is projective in $\sigma[M]$, by [3, Proposition 5.5] we have that

$$L_M(H_M(H_M K)) = (L_M H)_M(H_M K) \leq N_M(H_M K) = 0$$

Now, since $H_M(H_M K) \leq H_M K$, then

$$Ann_M(K) = Ann_M(H_M K) \leq Ann_M(H_M(H_M K))$$

Therefore $Ann_M(H_M K) = Ann_M(H_M(H_M K))$. Thus $L \leq Ann_M(K) = N$.

Now, let $P \leq M$ be a prime submodule of M such that $P < N$. We have that $N_M K = 0 \leq P$. So $K \leq P < N$. Hence $K_M K = 0$. Thus $K = 0$, a contradiction. It follows that N is a minimal prime submodule of M.

2 $\Rightarrow$ *3* : By hypothesis.

3 $\Rightarrow$ *1* : Suppose $N < K$ with K an annihilator submodule. Then

$$Ann_M(K)_M K = 0 \leq N$$

Since N is prime in M, then $Ann_M(K) \leq N < K$. By Proposition 2.2.8 $Ann_M(K) \cap K = 0$, hence $Ann_M(K) = 0$. Since K is an annihilator submodule, by Proposition 2.2.7, $K = Ann_M(Ann_M(K)) = Ann_M(0)$, a contradiction. □

Proposition 2.2.11. *Let M be projective in $\sigma[M]$ and semiprime. For $N \leq M$, if $N = Ann_M(U)$ with $U \leq M$ a uniform submodule, then N is a maximal annihilator submodule.*

Proof. Suppose that $N < K$ with K an annihilator submodule in M. Since $N = Ann_M(U)$ by Proposition 2.2.8, $K \cap U \neq 0$. By hypothesis U is uniform and thus $K \cap U \leq_e U$. Then

$$(K \cap U) \oplus Ann_M(U) \leq_e U \oplus Ann_M(U)$$

Now, notice that if $L \leq_{fi} M$, by Proposition 2.2.8 $(U \oplus Ann_M(U)) \cap L \neq 0$. So $0 \neq ((K \cap U) \oplus Ann_M(U)) \cap L \leq K$. Therefore, $K \cap L \neq 0$ and K intersects all fully invariant submodules of M . Since $K \cap Ann_M(K) = 0$ and $Ann_M(K) \leq_{fi} M$, then $Ann_M(K) = 0$. Thus, $K = Ann_M(Ann_M(K)) = Ann_M(0)$, a contradiction. $\square$

Next proposition will be applied in the main theorem (Theorem 2.4.10) of Section 2.4.

Proposition 2.2.12. *Let M be projective in $\sigma[M]$ and semiprime with finite uniform dimension. Then:*

1. *M has finitely many minimal prime submodules.*
2. *The number of annihilator submodules is finite.*
3. *M satisfies ACC on annihilator submodules.*

Proof. *1* : Let $U_1, .., U_n$ be uniform submodules of M such that $U_1 \oplus ... \oplus U_n \leq_e M$. By Propositions 2.2.10 and 2.2.11, $P_i := Ann_M(U_i)$ is a minimal prime submodule in M for each $1 \leq i \leq n$.

By Proposition 2.2.8, $(U_1 \oplus ... \oplus U_n) \cap Ann_M(U_1 \oplus ... \oplus U_n) = 0$ and $P_1 \cap ... \cap P_n \leq Ann_M(U_1 \oplus ... \oplus U_n) = 0$.

Now, if P is a minimal prime submodule of M, then

$$P_{1M} P_{2M} ..._M P_n \leq P_1 \cap ... \cap P_n = 0 \leq P$$

Hence, there exists j such that $P_j \leq P$. Thus $P = P_j$ for some $1 \leq j \leq n$.

2 : By Lemma 2.2.9.

3 : It is clear by *2*. $\square$

2.3 Large Primes

In this section, it will be given a generalization of prime submodules defined in Definition 2.1.21. This generalization was motivated by the commutative case when maximal ideals are prime and as consequence the Jacobson Radical is a semiprime ideal.

In the general context, left sided maximal ideals are not prime but it is well known that the Jacobson Radical is a semiprime ideal (Two sided) and there are examples of rings R where $Rad({}_RR) \neq Rad({}_RR_R)$. So, we want to extend $Spec(\Lambda^{fi}(M))$ in a such way that the maximal submodules are included.

In [22, Proposition 18] it is proved the following.

Proposition 2.3.1. *Suppose M is quasi-projective and let $N \in \Lambda^{fi}(M)$. Then N is prime in M if and only if M/N is a prime module.*

In the converse of this proposition it is not used that $N \in \Lambda^{fi}(M)$. So, we can write the following result.

Proposition 2.3.2. *Let M be quasi-projective and $N \leq M$. If M/N is a prime module then for any $L, K \in \Lambda^{fi}(M)$ such that $L_M K \leq N$, it follows that $L \leq N$ or $K \leq N$.*

Proposition 2.3.3. *Let M be projective in $\sigma[M]$. If $N, L \in \Lambda^{fi}(M)$, then $N_M L \in \Lambda^{fi}(M)$ and hence the product $-_M-$ is well restricted in $\Lambda^{fi}(M)$. Moreover $\Lambda^{fi}(M)$ is a right subquasi-quantale of $\Lambda(M)$.*

Proof. Let $N, L \in \Lambda^{fi}(M)$ and $g : M \to M$. Then

$$g(N_M L) = g\left(\sum\{f(N) | f \in Hom_R(M, L)\}\right) = \sum\{gf(N) | f \in Hom_R(M, L)\}$$

Since $L \in \Lambda^{fi}(M)$, $gf \in Hom_R(M, L)$. Hence

$$g(N_M L) \subseteq N_M L.$$

Now, if $N \in \Lambda^{fi}(M)$ then

$$N_M M = \sum\{f(N) | f \in Hom_R(M, M)\} \subseteq N,$$

but $N = Id_M(N) \leq N_M M$. Hence $N_M M = N$. Thus $\Lambda^{fi}(M)$ is a right quasi-quantale. $\square$

Following Definition A.1.17 we can re-write Proposition 2.3.2 as if M/N is a prime module with M projective in $\sigma[M]$ then N is a prime relative to $\Lambda^{fi}(M)$.

Definition 2.3.4. The prime elements of $\Lambda(M)$ relatives to $\Lambda^{fi}(M)$ will be called *large primes* of M. The set of large primes of M will be denoted by $LgSpec(M)$ and is called *The Large Spectrum* of M.

Remark 2.3.5. Let M be projective in $\sigma[M]$. Then

1. $Spec(\Lambda^{fi}(M)) \subseteq LgSpec(M)$.
2. If $\mathcal{M}$ is a maximal submodule of M then $\mathcal{M} \in LgSpec(M)$.
3. If $Rad(M) \neq M$ then it is semiprime in M.
4. $Nil_*(M) \leq Rad(M)$.

Proof. 1. It is clear.

2. Let $\mathcal{M}$ be a maximal submodule of M. Since $\mathcal{M}$ is maximal, $M/\mathcal{M}$ is a simple module. Hence $M/\mathcal{M}$ is a prime module. By Proposition 2.3.2, $\mathcal{M}$ is a large prime.

3. By definition of large prime.

4. Since every maximal submodule is a large prime submodule, $Rad(M)$ is semiprime in M. So $Rad(M)$ is an intersection of elements of $Spec(\Lambda^{fi}(M))$ by Proposition 2.1.36. This implies that $Nil_*(M) \leq Rad(M)$. □

Let M be an R-module. For each $N \in \Lambda^{fi}(M)$, there are two distinguished preradicals, α_N^M (we have already mentioned it) and ω_N^M, which are defined as follow:

$$\alpha_N^M(L) := \sum \{f(N) \mid f \in Hom_R(M, L)\}$$

and

$$\omega_N^M(L) := \bigcap \{f^{-1}(N) \mid f \in Hom_R(L, M)\},$$

for each $L \in R$-Mod.

Remark 2.3.6. [21, Proposition 5]. If N is a fully invariant submodule of M, then the following conditions are satisfied.

1. The preradicals α_N^M and ω_N^M have the property that $\alpha_N^M(M) = N$ and $\omega_N^M(M) = N$ respectively.

2. The class $\{r \in R-pr \mid r(M)=N\}$ is precisely the interval $[\alpha_N^M, \omega_N^M]$.

The reader can find more properties of these preradicals in [21], [22], [23] and [20, Proposition 1.3].

Definition 2.3.7. Let M and P be R-modules such that $P \leq M$. Define

$$\eta_P^M\colon R-Mod \to R-Mod$$

$$\eta_P^M(L) := \bigcap\{f^{-1}(P) | f \in Hom_R(L, M)\},$$

for each $L \in R$-Mod.

It is clear that η_P^M is a preradical. Also, it is clear that $\eta_P^M(M) \leq P$. When $P \in \Lambda^{fi}(M)$, it follows that $P = \eta_P^M(M)$. And in this case, $\omega_P^M = \eta_P^M$.

Remark 2.3.8. Let M be an R-module.

1. If $P, Q \leq M$ satisfy that $P \leq Q$, then $\eta_P^M \preceq \eta_Q^M$.
2. If $r \in R-pr$ and $P \leq M$ satisfy that $r(M) \leq \eta_P^M(M)$, then

$$r \preceq \omega_{r(M)}^M \preceq \eta_P^M \preceq \omega_{\eta_P^M(M)}^M.$$

Proposition 2.3.9. *Let $M \in R$-Mod and $P \leq M$. If η_P^M is a prime preradical i.e. $\eta_P^M \in Spec(R-pr)$ and $\eta_P^M(M) \neq M$, then $\eta_P^M(M) \in Spec(\Lambda^{fi}(M))$. Moreover, $\eta_P^M(M)$ is the largest element in $Spec(\Lambda^{fi}(M))$ which is contained in P.*

Proof. Let $N, L \in \Lambda^{fi}(M)$ such that $N_M L \leq \eta_P^M(M)$. This is, $(\alpha_N^M \cdot \alpha_L^M)(M) \leq \eta_P^M(M)$. By Remark 2.3.8.*1* it follows that $\alpha_N^M \cdot \alpha_L^M \preceq \eta_P^M$. Since η_P^M is a prime preradical, we get $\alpha_N^M \preceq \eta_P^M$ or $\alpha_L^M \preceq \eta_P^M$. Thus, $N \leq \eta_P^M(M)$ or $L \leq \eta_P^M(M)$. Therefore, $\eta_P^M(M) \in Spec(\Lambda^{fi}(M))$.

Finally, let $K \in \Lambda^{fi}(M)$ be a prime submodule of M such that $K \leq P$. Then, $\omega_K^M \preceq \eta_P^M$. Thus, applying M in the last inequality we obtain $K = \omega_K^M(M) \leq \eta_P^M(M) \leq P$. $\square$

From now on, M will be assumed projective in $\sigma[M]$, in order to have that $\Lambda^{fi}(M)$ is a right subquasi-quantale of $\Lambda(M)$.

Proposition 2.3.10. *Let M and P be R-modules such that $P \leq M$. Then, the following conditions are equivalent.*

1. *P is a large prime submodule of M.*
2. *η_P^M is a prime preradical.*

Proof. *1* $\Rightarrow$ *2*. From the hypothesis, it is clear that $P \neq M$. So, $\eta_P^M \neq 1$. Let $r, s \in R-pr$ such that $r \cdot s \preceq \eta_P^M$. Evaluating this in M and using [22, Proposition 14 (2)], we obtain $r(M)_M s(M) \leq \eta_P^M(M)$. Thus, $r(M)_M s(M) \leq \eta_P^M(M) \leq P$. so, by hypothesis, it follows that $r(M) \leq P$ or $s(M) \leq P$. And by Proposition 2.3.9, $\eta_P^M(M)$ is the largest fully invariant submodule of M which is contained in P. Hence, $r(M) \leq \eta_P^M(M)$ or $s(M) \leq \eta_P^M(M)$. Consequently, $r \preceq \eta_P^M$ or $s \preceq \eta_P^M$.

2 $\Rightarrow$ *1*. By hypothesis, η_P^M is a prime preradical, so in particular $P < M$. Let $N, L \in \Lambda^{fi}(M)$ such that $N_M L = \alpha_N^M(L) \leq P$. Notice that $N_M L = \alpha_N^M(L) = (\alpha_N^M \cdot \alpha_L^M)(M)$. Then, $\alpha_N^M \cdot \alpha_L^M \preceq \omega_{N_M L}^M$. Thus, by Remark 2.3.8.*2* we get $\alpha_N^M \cdot \alpha_L^M \preceq \eta_P^M$. Since η_P^M is prime, then $\alpha_N^M \preceq \eta_P^M$ or $\alpha_L^M \preceq \eta_P^M$. Applying M we obtain $N = \alpha_N^M(M) \leq \eta_P^M(M) \leq P$ or $L = \alpha_L^M(M) \leq \eta_P^M(M) \leq P$. By Remark 2.3.8.*2*, we conclude that $r \preceq \eta_P^M$ or $s \preceq \eta_P^M$. Hence, η_P^M is a prime preradical. □

Corollary 2.3.11. *If P is a large prime submodule of M, then $\eta_P^M(M)$ is the largest element in $Spec(\Lambda^{fi}(M))$ which is contained in P.*

Proof. It follows from Propositions 2.3.9 and 2.3.10. □

We have that $LgSpec(M)$ is a topological space by Proposition A.1.19.

Proposition 2.3.12. *Let M be projective in $\sigma[M]$. Then $Spec(\Lambda^{fi}(M))$ is a dense subspace of $LgSpec(M)$*

Proof. Let $\mathcal{U}(N) \neq \emptyset$ be an open set of $LgSpec(M)$ such that

$$\mathcal{U}(N) \cap Spec(\Lambda^{fi}(M)) = \emptyset.$$

Thus, the elements of $\mathcal{U}(N)$ are not fully invariant. Let $P \in \mathcal{U}(N)$. By Corollary 2.3.11, there exists $Q \in Spec(\Lambda^{fi}(M))$ such that $Q \subseteq P$, which is a contradiction. Thus $\mathcal{U}(N) = \emptyset$. So, $Spec(\Lambda^{fi}(M))$ is dense in $LgSpec(M)$. □

2.4 The Frame of Semiprime Submodules

Given an R-module M projective in $\sigma[M]$, the large spectrum of M

$$LgSpec(M) = \{P \in \Lambda^{fi}(M) | P \text{ is large prime in } M\}$$

is a topological space with open subsets

$$\mathcal{U}(N) = \{P \in LgSpec(M) | N \nsubseteq P\}$$

with $N \in \Lambda^{fi}(M)$. The closed subsets are given by

$$\nu(N) = \{P \in LgSpec(M) | N \subseteq P\}$$

with $N \in \Lambda^{fi}(M)$.

Let $\mathcal{O}(LgSpec(M))$ be the frame of open subsets of $LgSpec(M)$. Then we have a morphism of $\bigvee$-semilattices

$$\mathcal{U}\colon \Lambda^{fi}(M) \to \mathcal{O}(LgSpec(M))$$

given by $\mathcal{U}(N) = \{P \in LgSpec(M) \mid N \nsubseteq P\}$.

This morphism has a right adjunct $\mathcal{U}_*\colon \mathcal{O}(LgSpec(M)) \to \Lambda^{fi}(M)$ given by

$$\mathcal{U}_*(A) = \sum\{K \in \Lambda^{fi}(M) \mid \mathcal{U}(K) \subseteq A\}.$$

Proposition 2.4.1. *Let $N \in \Lambda^{fi}(M)$. Then $(\mathcal{U}_* \circ \mathcal{U})(N)$ is the largest fully invariant submodule of M contained in $\bigcap_{P\in\mathcal{V}(N)} P$.*

Proof. It follows by Proposition A.1.21. □

Remark 2.4.2. Note that if $\{P_i\}_I$ is a family of large prime submodules of M, then $\bigcap_I P_i$ is not necessary a fully invariant submodule of M but $\bigcap_I P_i$ satisfies the property that for every $L \in \Lambda^{fi}(M)$ such that $L_M L \leq \bigcap_I P_i$ then $L \leq \bigcap_I P_i$.

Proposition 2.4.3. *Let $\mu = \mathcal{U}_* \circ \mathcal{U}$ be as in Proposition 2.4.1. Let $N \in \Lambda^{fi}(M)$ then, $\mu(N) = N$ if and only if N is semiprime in M or $N = M$.*

Proof. Suppose that $\mu(N) = N$ and let $L \in \Lambda^{fi}(M)$ such that $L_M L \leq N$. We have that $\mu(N) \leq \bigcap_{P\in\mathcal{V}(N)} P$, so $L_M L \leq \bigcap_{P\in\mathcal{V}(N)} P$. By Remark 2.4.2,

$L \leq \bigcap_{P\in\mathcal{V}(N)} P$. Since L is a fully invariant submodule of M, by Proposition 2.4.1 $L \leq \mu(N)$. Thus $\mu(N)$ is semiprime in M.

Now suppose that N is semiprime in M. By Proposition 2.1.36,

$$N = \bigcap\{Q \mid N \leq Q \; Q \in Spec(\Lambda^{fi}(M))\}.$$

Since $Spec(\Lambda^{fi}(M))\subseteq LgSpec(M)$, then

$$\bigcap_{P\in\mathcal{V}(N)} P \subseteq \bigcap\{Q \in Spec(\Lambda^{fi}(M)) \mid N \leq Q\} = N.$$

This implies that $N = \mu(N)$. □

Corollary 2.4.4. *The closure operator $\mu\colon \Lambda^{fi}(M) \to \Lambda^{fi}(M)$ is a multiplicative pre-nucleus.*

Proof. By Theorem A.1.22. □

Remark 2.4.5. By definition μ is a closure operator and by Corollary 2.4.4 μ is a pre-nucleus then, μ is a nucleus in $\Lambda^{fi}(M)$.

Proposition 2.4.6. *Let M be an R-module and define*

$$SP(M) = \{N \in \Lambda^{fi}(M) \mid N \textit{ is semiprime }\} \cup \{M\}.$$

Then $SP(M)$ is a frame. Moreover, $SP(M) \cong \mathcal{O}(LgSpec(M))$ canonically as frames.

Proof. By Proposition 2.4.3, $\Lambda^{fi}(M)_\mu = SP(M)$. Since μ is a multiplicative nucleus then $SP(M)$ is a frame by Corollary A.1.14. □

Definition 2.4.7. Let L be a lattice. An element $1 \neq p \in L$ is called *$\wedge$-irreducible* if whenever $x \wedge y \leq p$ for any $x, y \in L$ then $x \leq p$ or $y \leq p$.

Given a frame F, *its points* is the set

$$pt(F) = \{p \in F \mid p \text{ is} \wedge \text{-irreducible}\}$$

Proposition 2.4.8. *Let M be an R-module. Then $pt(SP(M)) = Spec(\Lambda^{fi}(M))$*

Proof. It is clear that $Spec(\Lambda^{fi}(M)) \subseteq pt(SP(M))$. Now, let $P \in pt(SP(M))$ and $N, L \in \Lambda^{fi}(M)$ such that $N_M L \leq P$. By Proposition 2.4.4,

$$\mu(N) \cap \mu(L) = \mu(N \cap L) = \mu(N_M L) \leq \mu(P) = P.$$

Since $\mu(N), \mu(L) \in SP(M)$ then $N \leq \mu(N) \leq P$ or $L \leq \mu(L) \leq P$. □

Definition 2.4.9. Let F be a frame. It is said that F is *spatial* if it is isomorphic to $\mathcal{O}(X)$ for some topological space X. If each quotient frame of F is spatial, it is said that F is *totally spatial*.

For more information see [25] and [24].

Theorem 2.4.10. *Let M be projective in $\sigma[M]$. Suppose that for every fully invariant submodule $N \leq M$, the factor module M/N has finite uniform dimension. Then, the frame $SP(M)$ is totally spatial.*

Proof. Let $N \in SP(M)$. By [29, Lemma 9] M/N is projective in $\sigma[M/N]$. Since N is semiprime in M then M/N is a semiprime module (Definition 2.1.34) by Proposition 2.1.36. By hypothesis M/N has finite uniform dimension, so by Proposition 2.2.12 $Spec(\Lambda^{fi}(M/N))$ has finitely many minimal elements $(P_1/N), ..., (P_n/N)$ such that $0 = P_1/N \cap ... \cap P_n/N$.

Since $P_i/N \in Spec(\Lambda^{fi}(M/N))$ then $M/P_i \cong \frac{M/N}{P_i/N}$ is a prime module. Thus, by Proposition 2.1.23, $P_i \in Spec(\Lambda^{fi}(M))$ for all $1 \leq i \leq n$. Moreover, $N = P_1 \cap ... \cap P_n$. Since this intersection is finite, we can assume that it is irredundant. Thus by [18, Theorem 3.4], $SP(M)$ is totally spatial. □

For the definition of Krull dimension of a module M see [8].

Corollary 2.4.11. *Let M be projective in $\sigma[M]$. If M has Krull dimension, then $SP(M)$ is totally spatial.*

Proof. If M has Krull dimension then every factor module M/N so does. Now, if M/N has Krull dimension, by [8, Proposition 2.9] M/N has finite uniform dimension. So by Theorem 2.4.10 $SP(M)$ is totally spatial. □

Definition 2.4.12. A module M is *coatomic* if every submodule is contained in a maximal submodule of M.

Example 2.4.13. 1. Every finitely generated and every semisimple module is coatomic.

2. Semiperfect modules are coatomic.

3. If R is a left perfect ring, every left R-module is coatomic. (see [12])

If

$$Max(M) = \{\mathcal{M} < M \mid \mathcal{M} \text{ is a maximal submodule }\},$$

we have that $Max(M) \subseteq LgSpec(M)$. Notice that $Max(M)$ is a subspace of $LgSpec(M)$.

Suppose that M is coatomic. Then we have the adjunction

$$\begin{tikzcd} \Lambda^{fi}(M) \arrow[r, shift left, "m"] & \mathcal{O}(Max(M)) \arrow[l, shift left, "m_*"] \end{tikzcd}$$

defined as

$$m(N) = \{\mathcal{M} \in Max(M) \mid N \nsubseteq \mathcal{M}\},$$

and

$$m_*(A) = \sum\{K \in \Lambda^{fi}(M) \mid m(K) \subseteq A\}.$$

This adjunction can be factorized as

$$\begin{tikzcd} \Lambda^{fi}(M) \arrow[r, shift left, "\mathcal{U}"] \arrow[dr, shift left, "m"] & \mathcal{O}(LgSpec(M)) \arrow[l, shift left, "\mathcal{U}_*"] \arrow[d, shift left, "I"] \\ & \mathcal{O}(Max(M)) \arrow[ul, shift left, "m_*"] \arrow[u, shift left, "I_*"] \end{tikzcd} ,$$

where $I\colon \mathcal{O}(LgSpec(M)) \to \mathcal{O}(Max(M))$ is given by $I(A) = A \cap Max(M)$.

Following the proof of Proposition A.1.21, $(m_* \circ m)(N)$ is the largest fully invariant submodule contained in

$$\bigcap_{\mathcal{M}\in\mathcal{V}(N)\cap Max(M)} \mathcal{M}$$

Then

$$(m_* \circ m)(0) = \bigcap_{\mathcal{M}\in Max(M)} \mathcal{M} = Rad(M)$$

Note that the proof of Theorem A.1.22 can be applied to $\tau := m_* \circ m$, so τ is a multiplicative nucleus. Hence $R(M) := \Lambda^{fi}(M)_\tau$ is a frame by Corollary A.1.14. Moreover, $R(M)$ is a subframe of $SP(M)$.

Definition 2.4.14. Let M be an R-module. M is *co-semisimple* if every simple module in $\sigma[M]$ is M-injective.

If $M = {}_RR$, this is the definition of a *left V-ring*.

The following characterization of co-semisimple modules is given in [31, 23.1].

Proposition 2.4.15. *For an R-module M the following statements are equivalent:*

1. *M is co-semisimple.*
2. *Any proper submodule of M is an intersection of maximal submodules.*

Corollary 2.4.16. *Let M be a co-semisimple module. Then $\Lambda^{fi}(M)$ is a frame.*

Proof. Since M is co-semisimple then $SP(M) = \Lambda^{fi}(M)$. □

Following [19], an R-module M is called *duo* if every submodule is a fully invariant submodule.

Corollary 2.4.17. *Let M be a duo co-semisimple module then $\Lambda(M)$ is a frame.*

Proof. In this case $\Lambda(M) = \Lambda^{fi}(M)$. □

Chapter 3

Modules satisfying ACC on left Annihilators

3.1 ACC on left Annihilators

In this section, it will be introduced the concept of left annihilator in terms of an R-module. It will be studied the modules which satisfy ascending chain condition (ACC) on left annihilators.

The following definition was given first in [8].

Definition 3.1.1. Let M be an R-module. We call *left annihilator* in M a submodule which has the form

$$\mathcal{A}_X = \bigcap \{Ker(f) | f \in X\}$$

for some $X \subseteq End_R(M)$.

Definition 3.1.2. Let M be an R-module and N a fully invariant submodule of M. We define the *powers* of N as:

1. $N^0 = 0$
2. $N^1 = N$
3. $N^m = N_M N^{m-1}$

We say that N is nilpotent if there exists $n > 0$ such that $N^n = 0$.

Proposition 3.1.3. *Let M be projective in $\sigma[M]$. If M satisfies ACC on left annihilators, then $\mathcal{Z}(M)$ is nilpotent.*

Proof. Consider the descending chain

$$\mathcal{Z}(M) \geq \mathcal{Z}(M)^2 \geq \mathcal{Z}(M)^3 \geq ...$$

Then, we have the ascending chain

$$Ann_M(\mathcal{Z}(M)) \leq Ann_M(\mathcal{Z}(M)^2) \leq Ann_M(\mathcal{Z}(M)^3) \leq ...$$

Since M satisfies ACC on left annihilators, there exists $n \geq 0$ such that $Ann_M(\mathcal{Z}(M)^n) = Ann_M(\mathcal{Z}(M)^{n+1})$.

Suppose that $\mathcal{Z}(M)^{n+2} \neq 0$ i.e. $\mathcal{Z}(M)^2{}_M\mathcal{Z}(M)^n \neq 0$, so

$$0 \neq \mathcal{Z}(M)^2{}_M\mathcal{Z}^n = \sum\{f(\mathcal{Z}(M))|f : M \to \mathcal{Z}(M)\}_M\mathcal{Z}(M)^n$$

$$= \sum\{f(\mathcal{Z}(M))_M\mathcal{Z}(M)^n|f : M \to \mathcal{Z}(M)\}$$

then there exists $f : M \to \mathcal{Z}(M)$ such that $f(M)_M\mathcal{Z}(M)^n \neq 0$. Consider the set

$$\Gamma = \{Ker(f)|f : M \to \mathcal{Z}(M)\ f(M)_M\mathcal{Z}(M)^n \neq 0\}$$

By hypothesis Γ has maximal elements, let $f : M \to \mathcal{Z}(M)$ such that $Ker(f)$ is a maximal element in Γ.

Let $h : M \to \mathcal{Z}(M)$, by [5, Proposition 2.5] $Ker(h) \leq_e M$, so $Ker(h) \cap f(M) \neq 0$ i.e. there exists $0 \neq f(m)$ such that $h(f(m)) = 0$. Then $Ker(f) < Ker(h \circ f)$. Since $Ker(f)$ is a maximal element in Γ then $h(f(M))_M\mathcal{Z}(M)^n = 0$. Thus, $h(f(M))_M\mathcal{Z}(M)^n = 0$ for all $h : M \to \mathcal{Z}(M)$. Hence

$$0 = (f(M)_M\mathcal{Z}(M))_M\mathcal{Z}(M)^n = f(M)_M\mathcal{Z}(M)^{n+1}$$

so, $f(M) \leq Ann_M(\mathcal{Z}(M)) = Ann_M(\mathcal{Z}(M)^n)$, this is a contradiction. Thus $\mathcal{Z}(M)^{n+2} = 0$. □

Corollary 3.1.4. *Let M be projective in $\sigma[M]$ and $S = End_R(M)$. If M satisfies ACC on left annihilators, then the ideal $\Delta = \{f \in S|Ker(f) \leq_e M\}$ is nilpotent.*

Proof. Notice that if $N, L \leq M$ then

$$Hom_R(M, L)Hom_R(M, N) \leq Hom_R(M, N_M L)$$

so for all $k \geq 0$, $Hom_R(M, N)^k \leq Hom_R(M, N^k)$.

Note that $\Delta M \leq \mathcal{Z}(M)$ and $\Delta \subseteq Hom_R(M, \Delta M)$, so for all $n \geq 0$

$$\Delta^n \subseteq Hom_R(M, \Delta M)^n \subseteq Hom_R(M, \mathcal{Z}(M))^n \subseteq Hom_R(M, \mathcal{Z}(M)^n)$$

By Proposition 3.1.3 $\mathcal{Z}(M)$ is nilpotent, so there exists $m > 0$ such that $\Delta^m = 0$. Thus Δ is nilpotent. □

Corollary 3.1.5. *Let M be projective in $\sigma[M]$ and $S = End_R(M)$. Suppose that M satisfies ACC on left annihilators. If I is an ideal of S such that $\cap_{f\in I} \ker f \leq_e M$, then I is nilpotent.*

Proof. Since $\cap_{f\in I} \ker f \leq_e M$, then $\ker f \leq_e M$ for all $f \in I$. Thus $I \leq \Delta$. By Corollary 3.1.4 we have that Δ is nilpotent. So I is nilpotent □

Corollary 3.1.6. *Let M be projective in $\sigma[M]$ and $S = End_R(M)$. If M is retractable and satisfies ACC on left annihilators, then $\mathcal{Z}_r(S)$ is nilpotent, where $\mathcal{Z}_r(S)$ is the right singular ideal of S.*

Proof. We claim that $\mathcal{Z}_r(S) \leq \Delta$.

Let $\alpha \in \mathcal{Z}_r(S)$, then there exists an essential right ideal I of S such that $\alpha I = 0$. Hence $\alpha\left(\sum_{g\in I} g(M)\right) = 0$. Now let $0 \neq N \leq M$. Since M is retractable, then there exists a non zero morphism $\beta : M \to N$. Thus $\beta S \cap I \neq 0$. So there exists $h \in S$ such that $0 \neq \beta \circ h \in I$. Hence $0 \neq (\beta \circ h)(M) \leq N \cap \sum_{g\in I} g(M)$. Thus $\sum_{g\in I} g(M) \leq_e M$ which implies $Ker\alpha \leq_e M$. Therefore $\mathcal{Z}_r(S) \leq \Delta$.

The result follows from Corollary 3.1.4. □

In a ring R there are weaker concepts of nilpotency. Let us remember one of them.

Definition 3.1.7. Let R be a ring and I a proper left ideal. It is said I is *left T-nilpotent* if for any sequence $a_1, a_2, ... \in I$ there exists $n > 0$ such that $a_1 a_2 \cdots a_n = 0$

This definition can be generalized, in our context, as follows.

Definition 3.1.8. Let M be an R-module. A proper submodule N of M is *T_M-nilpotent* if for every sequence $f_1, f_2, ..., f_n, ... \in Hom_R(M, N)$ and any $a \in N$, there exists $n \geq 1$ such that $f_n f_{n-1} ... f_1(a) = 0$.

Remark 3.1.9. Let I be a left ideal of a ring R. Then I is left T-nilpotent if and only if I is T_R-nilpotent.

Proof. $\Rightarrow$. Let $f_1, f_2, ..., f_n, ... \in Hom_R(R, I)$ be a sequence and $a_0 \in I$. Each f_i has the form $f_i = (\_ \cdot a_i)$ with $a_i \in I$ for all $i > 0$. Hence, we have a sequence $a_0, a_1, a_2, ... \in I$. Since I is left T-nilpotent there exists $n > 0$ such that $a_0 a_1 \cdots a_{n-1} a_n = 0$, but this can be seen as

$$(\_ \cdot a_n)(\_ \cdot a_{n-1}) \cdots (\_ \cdot a_1)(a_0) = 0$$
$$= f_n f_{n-1} \cdots f_1(a_0)$$

Thus I is T_R-nilpotent.

$\Leftarrow$. It is analogous. □

Proposition 3.1.10. *Let M be projective in $\sigma[M]$ and retractable. Suppose that M satisfies ACC on left annihilators. If $N \leq M$ is T_M-nilpotent, then N is nilpotent.*

Proof. Suppose $N \leq M$ is T_M-nilpotent. Consider the chain $N \supseteq N^2 \supseteq N^3 \supseteq ...$ Then we have the ascending chain $Ann_M(N) \leq Ann_M(N^2) \leq Ann_M(N^3) \leq ...$

Since M satisfies ACC on left annihilators, there exists $k \geq 1$ such that

$$Ann_M\left(N^k\right) = Ann_M\left(N^{k+1}\right) = Ann_M\left(N^{k+2}\right) ...$$

Let $L = N^k$. If $L^2 = L_M L \neq 0$, then there exist $f : M \to L$ and $a \in L \leq N$ such that $f(a) \neq 0$. Hence $(Ra)_M L \neq 0$. If $(Ra)_M (L_M L) = 0$, then $Ra \leq Ann_M(L^2) = Ann_M(L)$. Contradiction. Thus $(Ra)_M (L_M L) \neq 0$. Since $(Ra)_M L = \sum \{f(Ra) \mid f : M \to L\}$, then there exists $f_1 : M \to L$ such that $f_1(Ra)_M L \neq 0$. So $(Rf_1(a))_M L \neq 0$. This implies that $(Rf_1(a)_M L)_M L \neq 0$. So there exists $f_2 : M \to L$ such that $f_2(Rf_1(a))_M L \neq 0$, and then $Rf_2(f_1(a))_M L \neq 0$. Continuing in this way, we have that $Rf_n f_{n-1} ... f_1(a)_M L \neq 0$ for all $n \geq 1$. Hence $f_n f_{n-1} ... f_1(a) \neq 0$ for all $n \geq 1$, contradiction. Hence $L^2 = 0$. So N is nilpotent. □

3.2 Semiprime Modules satisfying ACC on left Annihilators

In this section it will be proved two theorems that will show some of the semiprime module structure when these satisfy ACC on left annihilators.

One of the fundamental facts in this section is that every semiprime module with ACC on left annihilators has finitely many minimal prime submodules. With this, we will be able to make a bijective correspondence between minimal primes and some indecomposable injective modules.

Let us start with one of the theorems mentioned.

Theorem 3.2.1. *Let M be projective in $\sigma[M]$ and a semiprime module. If M satisfies ACC on left annihilators, then:*

1. *M has finitely many minimal prime in M submodules.*
2. *If P_1, P_2,..., P_n are the distinct minimal prime in M submodules, then $P_1 \cap P_2 \cap ... \cap P_n = 0$.*
3. *If $P \leq M$ is prime in M, then P is a minimal prime in M if and only if P is an annihilator submodule.*

Proof. 1. We will call "prime annihilator" a such annihilator submodule which is prime in M.

We shall first show that every annihilator submodule of M contains a finite product of prime annihilators. Suppose not. By hypothesis there is an annihilator submodule N which is maximal with respect to not contain a finite product of prime annihilators. In particular N is not prime in M. Thus by Proposition 2.1.36 there are fully invariant submodules L and K of M such that $N < L$, $N < K$ and $L_M K \leq N$. Since $Ann_M(N)_M N = 0$, then $Ann_M(N)_M(L_M K) = 0$. Since M is projective in $\sigma[M]$, then $(Ann_M(N)_M L)_M K = 0$. Therefore $K \leq Ann_M(Ann_M(N)_M L)$.

Remember that when M is semiprime $Ann^r_M(L) = Ann_M(L)$ for all $L \leq M$. Corollary 2.2.5.

We claim that $L_M[Ann_M(Ann_M(N)_M L)] \leq N$.

We have that $[Ann_M(N)_M L]_M [Ann_M(Ann_M(N)_M L)] = 0$, then $\sigma[M]$, then $Ann_M(N)_M(L_M[Ann_M(Ann_M(N)_M L)]) = 0$. Hence

$$L_M[Ann_M(Ann_M(N)_M L)] \leq Ann_M(Ann_M(N))$$

Since N is an annihilator submodule, by Proposition 2.2.7

$$L_M\left[Ann_M\left(Ann_M\left(N\right)_M L\right)\right] \leq N$$

Let $K' = \left[Ann_M\left(Ann_M\left(N\right)_M L\right)\right]$ then $L_M K' \leq N$ and $N < K'$.

In an analogous way, it can be proved that $L \leq Ann_M\left[K'_M Ann_M\left(N\right)\right] = L'$ and $L'_M K' \leq N$. Therefore K' and L' are annihilator submodules such that $N < K'$, $N < L'$, and $L'_M K' \leq N$. So L' and K' contain a finite product of prime annihilators. Hence N contains a finite product of prime annihilator. Contradiction. Thus every annihilator submodule of M contains a finite product of prime annihilators.

Since $0 = Ann_M\left(M\right)$, there exist prime annihilators P_1, P_2,..., P_n such that $\left(P_1\right)_M \left(P_2\right)_M \cdots_M P_n = 0$. If Q is a minimal prime in M, then $\left(P_1\right)_M \left(P_2\right)_M \ ... \ _M\left(P_n\right) = 0 \leq Q$. So there exists $P_j \leq Q$ for some $0 \leq j \leq n$. Then $P_j = Q$. Thus, the minimal prime in M submodules are some of $\{P_1, P_2, \ldots, P_n\}$. Moreover each P_i is an annihilator submodule for all $0 \leq i \leq n$.

2. Let $P_1, ..., P_n$ be the minimal prime in M submodules. Since $P_1 \cap ... \cap P_n \leq P_i$ for all $1 \leq i \leq n$ then $\left[P_1 \cap P_2 \cap, ..., \cap P_n\right]^n \leq \left(P_1\right)_M \left(P_2\right)_M \cdots_M \left(P_n\right) = 0$. Since M is semiprime $P_1 \cap P_2 \cap, ..., \cap P_n = 0$.

3. By (*1*) each minimal prime is an annihilator submodule. The converse is by Proposition 2.2.10. □

Note that if M does not satisfy ACC on left annihilators, Theorem 3.2.1 is not true. In order to see this, consider de following example:

Example 3.2.2. Let R be a ring such that $\{S_i\}_{i \in I}$ is an infinite family of non isomorphic simple R-modules and let $M = \bigoplus_{i \in I} S_i$. It is clear that M is projective in $\sigma\left[M\right]$. Now let $N \leq M$ such that $N_M N = 0$. Since M is semisimple then $N = 0$. Thus M is a semiprime module.

We claim that M does not satisfy ACC on left annihilators. In fact, let L be a submodule of M, then there exists K such that $M = K \oplus L$. If $\pi : M \to K$ is the canonical projection, then $Ker(\pi) = L$. Thus each submodule of M is the kernel of a morphism. Hence M does not satisfy ACC on left annihilators. Now since $S_i \ncong S_j$, then S_i is a minimal prime in M. So M has infinitely many minimal prime submodules in M .

Next definition was given in [7].

Definition 3.2.3. Let $N \in \sigma[M]$. A proper fully invariant submodule K of M is said to be *associated to* N if there exists $0 \neq L \leq N$ such that $Ann_M(L') = K$ for all non zero submodules $L' \leq L$.

If M is projective in $\sigma[M]$ and $P \leq M$ is associated to $N \in \sigma[M]$, then by [7, Lemma 1.16] P is prime in M.

Given $N \in \sigma[M]$, the set of submodules of M associated to N is denoted by $Ass_M(N)$.

Remark 3.2.4. Let M be projective in $\sigma[M]$ and semirpime. Let $U \leq M$ be a uniform submodule. By Proposition 2.2.11 and Proposition 2.2.10, $Ann_M(U) = P$ is a minimal prime in M. By [7, Proposition 4.4] $Ass_M(U) = \{P\}$. Moreover $Ass_M(U') = P$ for all $0 \neq U' \leq U$.

Lemma 3.2.5. *Let M be semiprime and projective in $\sigma[M]$. Suppose that M satisfies ACC on left annihilators. If P is a minimal prime in M and $Ann_M(P) = L$, then $P = Ann_M(L')$ for all $0 \neq L' \leq L$. Moreover $Ass_M(\widehat{L}) = Ass_M(L) = \{P\}$.*

Proof. By Theorem 3.2.1 P is an annihilator submodule, then

$$P = Ann_M(Ann_M(P)) = Ann_M(L)$$

by Proposition 2.2.7. Let $0 \neq L' \leq L$ and $Ann_M(L') = K$. Thus $P = Ann_M(L) \subseteq Ann_M(L') = K$. Suppose $P < K$. Since P is prime in M and $Ann_M(K)_M K = 0 \leq P$ then $Ann_M(K) \leq P$. So $Ann_M(K) \leq K$. Thus $Ann_M(K) = 0$ by Lemma 2.2.4.

Since K is an annihilator submodule

$$K = Ann_M(Ann_M(K)) = Ann_M(0) = M$$

Thus $M = K = Ann_M(L')$. By Lemma 2.1.44 $L' = 0$. Contradiction. Therefore $P = K = Ann_M(L')$. $\square$

Proposition 3.2.6. *Let M be semiprime and projective in $\sigma[M]$. Suppose that M satisfies ACC on left annihilators. If $P_1, P_2, ..., P_n$ are the minimal prime in M submodules then $\{N_1, N_2, ...N_n\}$ is an independent family, where $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$.*

Proof. By induction on n. If $n = 1$ it is obvious.

Suppose that $\{N_1, N_2, ..., N_{n-1}\}$ is an independent family. If

$$(N_1 \oplus N_2 \oplus ... \oplus N_{n-1}) \cap N_n \neq 0$$

there exists $x \in N_n$ such that $Rx \cong N_i' \leq N_i$ for some $1 \leq i \leq n-1$ by [10, 2.3.3 Projection Argument]. By Lemma 3.2.5 $P_n = Ann_M(Rx) = Ann_M(N_i') = Ann_M(N_i) = P_i$. Contradiction. Therefore

$$(N_1 \oplus N_2 \oplus ... \oplus N_{n-1}) \cap N_n = 0$$

. □

Lemma 3.2.7. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M satisfies ACC on left annihilators and $P_1, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$ for each $1 \leq i \leq n$ then N_i is a pseudocomplement of $\bigoplus\limits_{j \neq i} N_j$.*

Proof. It is enough to prove that N_1 is a pseudocomplemet of $\bigoplus_{i=2}^n N_i$. Let $L \leq M$ such that $L \cap (\bigoplus_{i=2}^n N_i) = 0$ and $N_1 \leq L$. Since $\bigoplus_{i=2}^n N_i$ is a fully invariant submodule then $(\bigoplus_{i=2}^n N_i)_M L \leq (\bigoplus_{i=2}^n N_i) \cap L = 0$. Since $N_{iM}L \leq (\bigoplus_{i=2}^n N_i)_M L = 0$ then $\bigoplus_{i=2}^n (N_{iM}L) = 0$. Hence $N_{iM}L = 0$ for all $2 \leq i \leq n$. So, by Proposition 2.2.8 and Theorem 3.2.1, $L \leq Ann_M(N_i) = Ann_M(Ann_M(P_i)) = P_i$ for all $2 \leq i \leq n$. Then $L \leq \bigcap_{i=2}^n P_i = Ann_M(P_1) = N_1$. Thus $L = N_1$. □

Corollary 3.2.8. *Let M be semiprime and projective in $\sigma[M]$. Suppose that M satisfies ACC on left annihilators. If $P_1, P_2, ..., P_n$ are the minimal prime in M submodules, then $\widehat{N_1} \oplus \widehat{N_2} \oplus ... \oplus \widehat{N_n} = \widehat{M}$, where $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$.*

Proof. By Lemma 3.2.7 $\bigoplus_{i=1}^n N_i \leq_e M$. So we have the result. □

Proposition 3.2.9. *Let M be semiprime and projective in $\sigma[M]$. Suppose that M satisfies ACC on left annihilators. If $P_1, P_2, ..., P_n$ are the minimal prime in M submodules and $N_i = Ann_M(P_i)$ has finite uniform dimension for each $1 \leq i \leq n$, then M has finite uniform dimension.*

Proof. By Lemma 3.2.7 $N_1 \oplus N_2 \oplus ... \oplus N_n \leq_e M$. Since N_i has finite uniform dimension for each $1 \leq i \leq n$, then M has finite uniform dimension. □

Remark 3.2.10. Let M be projective in $\sigma[M]$ and semiprime. Suppose that M satisfies ACC on left annihilators. Note that if M has only one minimal prime, then by Theorem 3.2.1.*2* it has to be zero.

Lemma 3.2.11. *Let M be semiprime and projective in $\sigma[M]$. Suppose that M satisfies ACC on left annihilators. If $P_1, P_2, ..., P_n$ are the minimal prime in M submodules, with $n > 1$ and P_i has finite uniform dimension for each $1 \leq i \leq n$, then M has finite uniform dimension.*

Proof. By Proposition 2.2.9 $Ann_M(P_i) = \underset{i\neq j}{\cap} P_j$. Hence $Ann_M(P_i)$ has finite uniform dimension. So by Proposition 3.2.8 we have that M has finite uniform dimension. □

Now, we want to establish a correspondence between minimal prime submodules of a module M and some indecomposable injective modules. These injective modules will be torsion free for a particular torsion theory in $\sigma[M]$. For general concepts and results in torsion theories in $\sigma[M]$ see [32].

Definition 3.2.12. Let τ be a hereditary torsion theory in $\sigma[M]$. A module $C \in \sigma[M]$ is called *τ-cocritical* if C is τ-torsion free and every proper factor module of C is τ-torsion. See [11].

Remark 3.2.13. Notice that every τ-cocritical module is uniform.

Lemma 3.2.14. *Let M be an R-module. If $C \in \sigma[M]$ is $\chi(M)$-cocritical, then there are submodules $C' \leq C$ and $M' \leq M$ such that C' is isomorphic to M'.*

Proof. Since C is $\chi(M)$-cocritical, then C is $\chi(M)$-torsion free, this implies that $Hom_R\left(C, \widehat{M}\right) \neq 0$. So there exists $0 \neq f : C \to \widehat{M}$. Since C is $\chi(M)$-cocritical and $M \leq_e \widehat{M}$, then there exists $C' \leq C$ such that C' embeds in M. Hence, there exists $M' \leq M$ such that $C' \cong M'$. □

Remark 3.2.15. If $C \in \sigma[M]$ is $\chi(M)$-cocritical, then by Lemma 3.2.14 there are $C' \leq C$ and $M' \leq M$ such that $C' \cong M'$. Hence M' is a uniform module, then $Ass_M(C') = Ass_M(M') = \{P\}$. Since $C' \leq_e C$, then $Ass_M(C) = \{P\}$.

Remark 3.2.16. Let τ_g be the hereditary torsion theory generated by the M-singular modules in $\sigma[M]$. If M is non M-singular, then $\tau_g = \chi(M)$. In fact, if M is non M-singular, then $\tau_g \leq \chi(M)$. Now if $\tau_g < \chi(M)$, then there exists a $\chi(M)$-torsion module $0 \neq N$ such that N is τ_g-torsion free. Thus $Hom_R\left(N, \widehat{M}\right) = 0$. By [32, Proposition 10.2] N is M-singular. Thus $N \in \mathbb{T}_{\tau_g}$ a contradiction. So $\tau_g = \chi(M)$.

Let $Spec^{Min}(M)$ denote the set of minimal prime in M submodules and $\mathcal{E}_\tau(M)$ a complete set of representatives of isomorphism classes of indecomposable τ-torsion free injective modules in $\sigma[M]$.

Theorem 3.2.17. *Let M be semiprime and projective in $\sigma[M]$. If M satisfies ACC on left annihilators and each $0 \neq N \leq M$ contains an uniform submodule, then there is a bijective correspondence between $\mathcal{E}_{\chi(M)}(M)$ and $Spec^{Min}(M)$.*

Proof. Since M is semiprime and M satisfies ACC on left annihilators then, by [8, Proposition 3.4] M is non M-singular. Thus by Remark 3.2.16 $\tau_g = \chi(M)$. Let $E \in \mathcal{E}_{\chi(M)}(M)$. Since E is a uniform $\chi(M)$-torsion free module then it is $\chi(M)$-cocritical. By Remark 3.2.15 $Ass_M(E) = \{P\}$ with $P \in Spec^{Min}(M)$, so we define

$$\Psi : \mathcal{E}_{\chi(M)}(M) \to Spec^{Min}(M)$$

as $\Psi(E) = P$.

We claim that Ψ is bijective. Suppose that $\Psi(E_1) = \Psi(E_2) = P$. Since E_1 and E_2 are $\chi(M)$-cocritical by Lemma 3.2.14 there exist $C_1' \leq E_1$, $C_2' \leq E_2$ and $M_1, M_2 \leq M$ such that $C_1' \cong M_1$ and $C_2' \cong M_2$. Hence, by Remark 3.2.15 $Ann_M(C_1') = Ann_M(M_1) = P = Ann_M(M_2) = Ann_M(C_2')$.

Consider $(M_1 + P)/P$ and $(M_2 + P)/P$. By [7, Proposition 2.2 and Proposition 2.7] $\chi((M_1 + P)/P) = \chi(M/P) = \chi((M_2 + P)/P)$. Since $Ann_M(M_1) = Ann_M(M_2) = P$, by Lemma 2.2.4 $M_1 \cap P = 0$ and $M_2 \cap P = 0$. Therefore $(M_1 + P)/P \cong M_1$ and $(M_2 + P)/P \cong M_2$. Thus $\chi(M_1) = \chi(M/P) = \chi(M_2)$, In particular $Hom_R\left(M_1, \widehat{M_2}\right) \neq 0$. Since M_1 and M_2 are τ_g-cocritical, then there exists $N_1 \leq M_1$ such that $N_1 \hookrightarrow M_2$. Since M_1 and M_2 are uniform modules, $\widehat{M_1} \cong \widehat{N_1} \cong \widehat{M_2}$. Thus $E_1 = \widehat{C_1'} \cong \widehat{M_1} \cong \widehat{M_2} \cong \widehat{C_2'} = E_2$. So Ψ is injective.

Now let $P \in Spec^{Min}(M)$. Since M satisfies ACC on left annihilators, then by Theorem 3.2.1 $P = Ann_M(N)$ for some $0 \neq N \leq M$. By hypothesis there exists a uniform module U such that $U \leq N$. Thus $P = Ann_M(N) \leq Ann_M(U)$. By Proposition 2.2.11 and Proposition 2.2.10 $P = Ann_M(U)$. Moreover $Ass_M(U) = \{P\} = Ass_M\left(\widehat{U}\right)$. Since M is $\chi(M)$-torsion free and U is uniform, then $\widehat{U} \in \mathcal{E}_{\chi(M)}(M)$. Thus $\Psi\left(\widehat{U}\right) = P$. □

Corollary 3.2.18. *Let R be a semiprime ring satisfying ACC on left annihilators. Suppose that each left ideal $0 \neq I$ contains a uniform left ideal. Then there is a bijective correspondence between the set of representatives of isomorphism classes of indecomposable non singular injective R-modules and the set of minimal prime ideals of R.*

Note that in last Corollary the condition that every non zero left ideal contains a uniform left ideal is necessary. In order to see this, consider the following example:

Example 3.2.19. Let R be a ring such that R is a right Ore domain but not left (see [28, pag. 53]). In [11] is proved that there are not $\chi(R)$-cocritical left R-modules. Since R is a domain, R is non singular. Thus $\chi(R) = \tau_g$. Moreover R is a prime ring and R clearly satisfies ACC on left annihilators. On the other hand if U is a uniform τ_g-torsion free module, then U is τ_g-cocritical, but this is not possible. Thus there are no uniform τ_g-torsion free modules. Hence $\mathcal{E}_{\tau_g}(R) = \emptyset$. Since R is a domain, $Spec^{Min}(R) = \{0\}$.

Chapter 4

Goldie Modules

This chapter is the main part of this thesis. Here, it will be presented the concept of Goldie module as a generalization of Goldie ring.

In the first section we will develop the tools about semiprime Goldie modules in order to prove a generalization of Goldie's Theorem.

Second section is concerned to study the structure of semiprime Goldie modules.

In this two section, it will be applied the previous results about semiprime modules and modules which satisfy ACC on left annihilators.

4.1 Semiprime Goldie Modules

Recall that an element c in a ring R is *left regular* (resp. *right regular*) if whenever $ca = 0$ with $a \in R$ implies $a = 0$ (resp. if $ac = 0$ implies $a = 0$). An element $c \in R$ is said to be regular if it is both left and right regular.

Since this section is concerned to show some generalizations of Goldie's Theorem, I include it here for reader's convenience. This version of Goldie's Theorem was taken from [16, Theorem 11.13].

Definition 4.1.1. Let R be a ring. It is said that R is a *left Goldie ring* if R has finite uniform dimension and has ACC on left annihilators.

Theorem 4.1.2 (Goldie's Theorem)**.** *For any ring R the following are equivalent:*

1. *R has a semisimple classical left quotient ring.*

2. *R is semiprime left Goldie.*

3. *R is semiprime with finite uniform dimension and has ACC on left annihilators of elements.*

4. *R is semiprime, left nonsingular with finite uniform dimension.*

5. *For any left ideal I of R, $I \leq_e R$ if and only if I contains a regular element.*

Notice that if $c \in R$ is a left regular element and $I \leq R$ is a left ideal such that if $c \in I$, then there exists a monomorphism from $Rc \cong R$ to I.

The following definition was taken from [27].

Definition 4.1.3. Let M be an R-module. M is *essentially compressible* if for every essential submodule $N \leq_e M$ there exists a monomorphism $M \to N$.

Lemma 4.1.4. *Suppose that M is projective in $\sigma[M]$. If $N \in \sigma[M]$ is essentially compressible, then $Ann_M(N)$ is a semiprime submodule of M.*

Proof. Let $L \leq M$ be a fully invariant submodule of M such that $L_M L \leq Ann_M(N)$. Put
$$\Gamma = \{K \leq N | L_M K = 0\}$$
Then $\Gamma \neq \emptyset$ and by Zorn's Lemma there exists a maximal independent family $\{K_i\}_I$ in Γ. Notice that $\bigoplus_I K_i \in \Gamma$ because

$$L_M \bigoplus_I K_i = \bigoplus_I L_M K_i = 0$$

Let $0 \neq A \leq N$ be a submodule. Since $(L_M L)_M A = 0$ then $L_M A \in \Gamma$.

If $L_M A = 0$ then $A \in \Gamma$ and $A \cap \bigoplus_I K_i \neq 0$ because $\{K_i\}_I$ is a maximal independent family in Γ.

Now, if $L_M A \neq 0$ we also have $(L_M A) \cap \bigoplus_I K_i \neq 0$ and $(L_M A) \cap \bigoplus_I K_i \leq A \cap \bigoplus_I K_i$. Thus $\bigoplus_I K_i \leq_e N$.

By hypothesis there exists a monomorphism $\theta : N \to \bigoplus_I K_i$. Then

$$\theta(L_M N) \leq L_M \bigoplus_I K_i = 0$$

and hence $L_M N = 0$. Thus $L \leq Ann_M(N)$. □

Proposition 4.1.5. *Let M be projective in $\sigma[M]$. If $N \in \sigma[M]$ is an M-singular module, then $Ker(f) \leq_e M$ for all $f \in Hom_R(M, N)$.*

Proof. Let $f \in Hom_R(M, N)$. Since N is M-singular, there exists an exact sequence

$$0 \longrightarrow K \xrightarrow{i} L \xrightarrow{\pi} N \longrightarrow 0$$

in $\sigma[M]$ with $K \leq_e L$. Since M is projective in $\sigma[M]$, there exists $\hat{f} : M \to L$ such that $\pi \hat{f} = f$:

$$\begin{tikzcd} & M \arrow[dl, "\hat{f}"'] \arrow[d, "f"] & \\ L \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

As $K \leq_e L$, then $\hat{f}^{-1}(K) \leq_e M$. Then

$$f(\hat{f}^{-1}(K)) = \pi(\hat{f}(\hat{f}^{-1}(K))) \leq \pi(K) = 0.$$

Therefore, $\hat{f}^{-1}(K) \leq Ker(f)$ and hence $Ker(f) \leq_e M$. $\square$

Proposition 4.1.6. *Let M be projective in $\sigma[M]$. If M is essentially compressible then M is non M-singular.*

Proof. Suppose $\mathcal{Z}(M) \neq 0$. If $\mathcal{Z}(M) \leq_e M$, then there exists a monomorphism $\theta : M \to \mathcal{Z}(M)$, by Proposition 4.1.5 $Ker\theta \leq_e M$, a contradiction. Therefore $\mathcal{Z}(M)$ has a pseudocomplement K in M and thus $\mathcal{Z}(M) \oplus K \leq_e M$. Hence, there exists a monomorphism $\theta : M \to \mathcal{Z}(M) \oplus K$. Let $\pi : \mathcal{Z}(M) \oplus K \to \mathcal{Z}(M)$ be the canonical projection, then $Ker(\pi\theta) \leq_e M$ and so $Ker(\pi\theta) = \theta^{-1}(Ker\pi) = \theta^{-1}(K) \leq_e M$. But $\mathcal{Z}(M) \cap \theta^{-1}(K) = 0$, a contradiction. Thus $\mathcal{Z}(M) = 0$. $\square$

Lemma 4.1.7. *Let M be an R-module with finite uniform dimension. Then, for every monomorphism $f : M \to M$, $Im(f) \leq_e M$.*

Proof. Let $f : M \to M$ be a monomrfism. If the uniform dimension of M is n, ($Udim(M) = n$) and there exists $K \leq M$ such that $f(M) \cap K = 0$, then $Udim(f(M) \oplus K) \leq n + 1$, a contradiction. $\square$

Next definition is the principal concept in this chapter

Definition 4.1.8. Let M be an R-module. We say that M is a *Goldie module* if it satisfies ACC on left annihilators and has finite uniform dimension.

Example 4.1.9.
- Every left Goldie ring is a Goldie module over itself.
- Every finite direct sum of simple modules is a Goldie module.
- Every noetherian module is a Goldie module.

Now, it will be presented a Goldie's Theorem version in the context of $\sigma[M]$.

Theorem 4.1.10. *Let M be projective in $\sigma[M]$ with finite uniform dimension. The following conditions are equivalent:*

1. *M is semiprime and non M-singular*
2. *M is semiprime and satisfies ACC on left annihilators*
3. *Let $N \leq M$, then $N \leq_e M$ if and only if there exists a monomorphism $f : M \to N$.*

Proof. *1* $\Rightarrow$ *2* : Since M is non M-singular and has finite uniform dimension then, by [8, Proposition 3.6] M satisfies ACC left on annihilators. This proves *2*.

2 $\Rightarrow$ *3* : Let $N \leq M$. Suppose that $N \leq_e M$. Since M is semiprime with uniform dimension and satisfies ACC on left annihilators, then M is essentially compressible by [8, Proposition 3.13]. Now, if $f : M \to N$ is a monomorphism then $N \leq_e M$ by Lemma 4.1.7.

3 $\Rightarrow$ *1* : It follows from Lemma 4.1.4 and Proposition 4.1.6. □

Remark 4.1.11. In [8, Proposition 3.13] M should be a generator of $\sigma[M]$, but by Lemma 2.1.44 this hypothesis is not necessary.

Definition 4.1.12. Let M be an R-module. A submodule $V \leq M$ is monoform if whenever $f : V \to M$ then $f = 0$ or f is a monomorphism.

M has enough monoforms if every submodule of M contains a monoform.

Corollary 4.1.13. *Let M be projective in $\sigma[M]$ and semiprime. Then, M has finite uniform dimension and enough monoforms if and only if M is a Goldie module.*

Proof. $\Rightarrow$: Since M is semiprime with finite uniform dimension and enough monoforms, then M is non M-singular by [8, Proposition 3.8] By Theorem 4.1.10, M is a Goldie module.

$\Leftarrow$: If M is a Goldie module, M has finite uniform dimension and by Theorem 4.1.10 M is non M-singular. Hence the uniform submodules of M are monoform. Since M has finite uniform dimension every submodule of M contains a uniform, hence every submodule contains a monoform. $\square$

For the definition of M-Gabriel dimension see [7] section 4.

Corollary 4.1.14. *Let M be projective in $\sigma[M]$ with finite uniform dimension. If M is a semiprime module and has M-Gabriel dimension, then M is a Goldie module.*

Proof. Let $N \leq M$. Since M has M-Gabriel dimension, by [7, Lemma 4.2], N contains a cocritical submodule L. Then L is monoform. By Corollary 4.1.13 M is a Goldie module. $\square$

For definition of Krull dimension of modules see [8]

Corollary 4.1.15. *Let M be projective in $\sigma[M]$ and semiprime with Krull dimension. Then M is a semiprime Goldie module.*

Proof. Since M has Krull dimension, M has finite uniform dimension and enough monoforms. By Proposition 4.1.13 M is a Goldie module. $\square$

Next propositions give some descriptions of the M-singular submodule.

Proposition 4.1.16. *Suppose that M is progenerator of $\sigma[M]$. Let $N \in \sigma[M]$, then*

$$\mathcal{Z}(N) = \sum\{f(M)|f : M \to N \; ker(f) \leq_e M\}.$$

Proof. By definition of M-singular module, it is clear that $\sum\{f(M)|f : M \to N \; ker(f) \leq_e M\} \leq \mathcal{Z}(N)$. Now, let $n \in \mathcal{Z}(N)$ and consider $Rn \leq \mathcal{Z}(N)$. Since $Rn \in \sigma[M]$ there exists a natural number t and an epimorphism $\rho : M^t \to Rn$. Suppose that $(m_1, .., m_t)$ is such that $\rho(m_1, ..., m_t) = n$. If $j_i : M \to M^t$ are the inclusions $(i = 1, ..., t)$, then by Proposition 4.1.5 $Ker(\rho \circ j_i) \leq_e M$. Thus, $n = \sum_{i=1}^{t} \rho \circ j_i(m_i) \in \sum\{f(M)|f : M \to N \; ker(f) \leq_e M\}$. $\square$

Remark 4.1.17. Let M be an R-module and consider $\tau_g \in M - tors$. If M is non M-singular, by Remark 3.2.13 $\chi(M) = \tau_g$. Let t_{τ_g} be the preradical associated to τ_g. Then

$$t_{\tau_g}(N) = \sum\{S \leq N|S \in \mathbb{T}_{\tau_g}\} = \sum\{S \leq N|S \; is \; M - singular\} = \mathcal{Z}(N).$$

Proposition 4.1.18. *Suppose M is progenerator of $\sigma[M]$. If M is semiprime Goldie, then*

$$\mathcal{Z}(N) = \sum f(M)$$

where the sum is over the $f : M \to N$ such that there exists a monomorphism $\alpha \in End_R(M)$ with $\alpha(M) \leq_e M$ and $f\alpha = 0$.

Proof. Let $N \in \sigma[M]$. By Proposition 4.1.16

$$\mathcal{Z}(N) = \sum\{f(M)|f : M \to N \; ker(f) \leq_e M\}.$$

If $f : M \to N$ with $Ker(f) \leq_e M$, by Theorem 4.1.10 there exists a monomorphism $\alpha : M \to Ker(f)$. We have that $f\alpha = 0$ and by Lemma 4.1.7 $\alpha(M) \leq_e (M)$.

Let $f : M \to N$ be such that there exists $\alpha : M \to M$ with $f\alpha = 0$ and $\alpha(M) \leq_e (M)$. Then $\alpha(M) \leq Ker(f)$. Therefore $Ker(f) \leq_e (M)$. $\square$

Remark 4.1.19. Let R be a ring such that R-Mod has an infinite set of non-isomorphic simples modules. Consider $M = \bigoplus_I S_i$, I an infinite set, such that S_i is a simple module for all $i \in I$ and with $Si \ncong S_j$ if $i \neq j$. This module does not have finite uniform dimension and, in M-tors, $\tau_g = \chi$. Then, if $N \in \sigma[M]$

$$t_{\tau_g}(N) = \mathcal{Z}(N) = \sum f(M)$$

where the sum is over the $f : M \to N$ such that there exists a monomorphism $\alpha \in End_R(M)$ with $\alpha(M) \leq_e M$ and $f\alpha = 0$.

Some authors have studied generalizations of Goldie's Theorem in the context of modules. They have used different methods and made different assumptions. For example, see [30] and [32].

Now, I want to unify these different versions. So, let us give some definitions used by the other authors.

The following definition appears in [2].

Definition 4.1.20. A module M is *weakly compressible* if for any nonzero submodule N of M, there exists $f : M \to N$ such that $f \circ f \neq 0$.

Remark 4.1.21. Notice that if M is weakly compressible then M is a semiprime module. The converse holds if M is projective in $\sigma[M]$.

Next definition was taken from [13].

Definition 4.1.22. A module M is a *semiprojective* module if $I = Hom(M, IM)$ for any cyclic right ideal I of $End_R(M)$.

For other characterizations of semiprojective modules see [31].

Proposition 4.1.23. *Let M be projective in $\sigma[M]$ and retractable. Then, $S := End_R(M)$ is semiprime if and only if M is semiprime.*

Proof. $\Rightarrow$: Corollary 2.1.43.

$\Leftarrow$: If M is semiprime, since M is projective in $\sigma[M]$ then M is weakly compressible and semiprojective. Then, by [13, Theorem 2.6] S is semiprime. □

Lemma 4.1.24. *Let M be projective in $\sigma[M]$ and retractable. M is non M-singular if and only if $Hom_R(M/N, M) = 0$ for all $N \leq_e M$.*

Proof. $\Rightarrow$: If $N \leq_e M$ then M/N is M-singular, then $Hom_R(M/N, M) = 0$.

$\Leftarrow$: Suppose $\mathcal{Z}(M) \neq 0$. Since M is retractable there exists $0 \neq f : M \to \mathcal{Z}(M)$. By Proposition 4.1.7 $Ker(f) \leq_e M$, so there exists a non zero morphism form $M/Ker(f) \to M$. □

For a retractable R-module M, [32, 11.6] gives necessary and sufficient conditions in order to $T := End_R(\widehat{M})$ being semisimple, left artinian, and being the classical left quotient ring of $S = End_R(M)$. Also, in [13, Corollary 2.7], the authors give necessary and sufficient conditions for a semiprojective module M in order to S is a semiprime right Goldie ring. We give an extension of these results.

Theorem 4.1.25. *Let M be projective in $\sigma[M]$, $S = End_R(M)$ and $T = End_R(\widehat{M})$. The following conditions are equivalent:*

1. *M is a semiprime Goldie module.*
2. *T is semisimple right artinian and is the classical right quotient ring of S.*
3. *S is a semiprime right Goldie ring.*
4. *M is weakly compressible with finite uniform dimension, and for all $N \leq_e M$, $Hom_R(M/N, M) = 0$.*

Proof. *1* $\Rightarrow$ *2* : By Proposition 4.1.23, S is a semiprime ring. Since M is a Goldie module, then M is non M-singular with finite uniform dimension, hence by [31, Proposition 11.6], T is right semisimple and is the classical right quotient ring of S.

2 $\Rightarrow$ *3* : By [16, Theorem 11.13], S is a semiprime right Goldie ring .

3 $\Rightarrow$ *4* : By [13, Corollary 2.7].

4 $\Rightarrow$ *1* : Since M is weakly compressible then M is semiprime. By Lemma 4.1.24 M is non M-singular. Thus, by Theorem 4.1.10 M is a Goldie module. □

Corollary 4.1.26. *Let M be projective in $\sigma[M]$, $S = End_R(M)$ and $T = End_R(\widehat{M})$. The following conditions are equivalent:*

1. *M is a prime Goldie module.*
2. *T is simple right artinian and is the classical right quotient ring of S.*
3. *S is a prime right Goldie ring.*
4. *Given nonzero submodules N, K of M there exists a morphism $f : M \to N$ such that $K \nsubseteq Ker(f)$. M has finite uniform dimension and for all $N \leq_e M$, $Hom(M/N, M) = 0$.*

Proof. *1* $\Rightarrow$ *2* : By Proposition 4.1.25, S is a semiprime ring and T is right semisimple and the classical right quotient ring of S. Let $0 \neq I \leq T$ be an ideal. Since T is semisimple, there exits an ideal $J \leq T$ such that $T = I \oplus J$. Put $M_1 = I\widehat{M}$ and $M_2 = J\widehat{M}$. Then M_1 and M_2 are fully invariant submodules of $\widehat{M}$ and $M_1 \cap M_2 = 0$ since $I \cap J = 0$. Consider $M_1 \cap M$ and $M_2 \cap M$. If $f \in S$, then there exists $\hat{f} \in T$ such that $f = \hat{f}|_M$. Let $x \in M_1 \cap M$. Then $f(x) = \hat{f}(x) \in M_1 \cap M$ since M_1 is a fully invariant submodule of $\widehat{M}$. Thus $M_1 \cap M$ is a fully invariant submodule of M. In the same way, $M_2 \cap M$ is fully invariant in M. Since $(M_1 \cap M) \cap (M_2 \cap M) = 0$, then $(M_1 \cap M)_M(M_2 \cap M) = 0$. Hence $M_1 \cap M = 0$ or $M_2 \cap M = 0$ because M is prime. On the other hand, $M \leq_e \widehat{M}$ and so $M_1 = 0$ or $M_2 = 0$. Since $0 \neq I$, then $M_2 = 0$. Thus $J = 0$, and it follows that T is a simple ring.

2 $\Rightarrow$ *3* : By [16, Corollary 11.16], S is a prime right Goldie ring.

3 $\Rightarrow$ *4* : Let N, K be nonzero submodules of M, if $K \subseteq Ker(f)$ for all $f : M \to N$ then $0 = Hom_R(M, N)Hom(M, K) \leq S$. Then $Hom_R(M, N) = 0$ or $Hom_R(M, K) = 0$. By retractability, $N = 0$ or $K = 0$, a contradiction.

4 $\Rightarrow$ *1* : It is clear. □

4.2 On the Structure of Goldie Modules

In this section will be proved some results about the structure of semiprime Goldie modules. In fact, many assertions concern to give decompositions of the M-injective hull of a semiprime Goldie module M.

For a good develop of the theory in this section we need to do the following convention:

Notation: Let X be an R-module and $K \in \sigma[X]$. We will denote by $E^{[X]}(K)$ the injective hull of K in $\sigma[X]$. When $X = K$ we write $E^{[X]}(X) = \widehat{X}$.

Since a Goldie module satisfies ACC on left annihilators, let us start with some consequences of the results in Chapter 3.

Theorem 4.2.1. *Let M be semiprime and projective in $\sigma[M]$. Suppose that M is a Goldie Module and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$, then there exist indecomposable injective modules E_1, E_2, ... , E_n such that $\widehat{M} \cong E_1^{k_1} \oplus E_2^{k_2} \oplus ... \oplus E_n^{k_n}$ and $Ass_M(E_i) = \{P_i\}$.*

Proof. Since N_i has finite uniform dimension, then there are uniform submodules $U_{i_1}, U_{i_2}, \ldots, U_{i_{k_i}}$ of N_i, such that $U_{i_1} \oplus U_{i_2} \oplus ... \oplus U_{i_{k_i}} \leq_e N_i$. Thus $E^{[M]}(U_{i_1}) \oplus E^{[M]}(U_{i_2}) \oplus ... \oplus E^{[M]}(U_{i_{k_i}}) = E^{[M]}(N_i)$. Now by Lemma 3.2.5 $Ann_M\left(U_{i_j}\right) = P_i$ for all $1 \leq j \leq k_i$. Thus $Ass_M\left(U_{i_j}\right) = Ass_M\left(E^{[M]}(U_{i_j})\right) = \{P_i\}$ for all $1 \leq j \leq k_i$. Hence, by Theorem 3.2.17 $E_i = E^{[M]}(U_{i_1}) \cong E^{[M]}(U_{i_2}) \cong .... \cong E^{[M]}(U_{i_{k_i}})$. So $E^{[M]}(N_i) \cong E_i^{k_i}$ for $1 \leq i \leq n$. Therefore, by Corollary 3.2.8 $\widehat{M} \cong E_1^{k_1} \oplus E_2^{k_2} \oplus ... \oplus E_n^{k_n}$.

Notice that $E_i \ncong E_j$ for $i \neq j$. $\square$

Proposition 4.2.2. *Let M be semiprime and projective in $\sigma[M]$. Suppose that M is a Goldie Module and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules, then $E^{[M]}(N_i)$ is a fully invariant submodule of $\widehat{M}$ where $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$.*

Proof. We claim that $Hom_R\left(E^{[M]}(N_i), E^{[M]}(N_j)\right) = 0$ if $i \neq j$. By the proof of Theorem 4.2.4 we have that $E^{[M]}(U_{i_1}) \oplus E^{[M]}(U_{i_2}) \oplus ... \oplus E^{[M]}(U_{i_{k_i}}) = E^{[M]}(N_i)$ where $U_{i_1}, U_{i_2}, \ldots, U_{i_{k_i}}$ are uniform submodules of N_i; analogously $E^{[M]}(U_{j_1}) \oplus E^{[M]}(U_{j_2}) \oplus ... \oplus E^{[M]}(U_{j_{k_j}}) = E^{[M]}(N_j)$ where $U_{j_1}, U_{i_{j_2}}, \ldots, U_{j_{k_j}}$ are uniform submodules of N_j. Let $0 \neq f \in Hom_R\left(E^{[M]}(N_i), E^{[M]}(N_j)\right)$, then there exist i_r and j_t such that the restriction $f|_{E^{[M]}(U_{i_r})} : E^{[M]}(U_{i_r}) \to E^{[M]}(U_{j_t})$ is non-zero.

Since M is a semiprime Goldie module, it is non M-singular, so $\widehat{M}$ is non M-singular. This implies that $f|_{E^{[M]}(U_{i_r})}$ is a monomorphism, so $Ass_M\left(E^{[M]}(U_{i_r})\right) = Ass_M\left(E^{[M]}(U_{j_t})\right)$. But

$$Ass_M\left(E^{[M]}(U_{i_r})\right) = Ass_M\left(E^{[M]}(N_i)\right) = P_i$$

and

$$Ass_M\left(E^{[M]}(U_{j_t})\right) = Ass_M\left(E^{[M]}(N_j)\right) = P_j$$

Contradiction. Thus $Hom_R\left(E^{[M]}(N_i), E^{[M]}(N_j)\right) = 0$.

Now let $g : \widehat{M} \to \widehat{M}$, by Corollary 3.2.8 $\widehat{M} = E^{[M]}(N_1) \oplus ... \oplus E^{[M]}(N_n)$. Since $Hom_R\left(E^{[M]}(N_i), E^{[M]}(N_j)\right) = 0$ for $i \neq j$, then $g\left(E^{[M]}(N_i)\right) \leq E^{[M]}(N_i)$. Thus $E^{[M]}(N_i)$ is a fully invariant submodule of $\widehat{M}$. □

Theorem 4.2.3. *Let M be projective in $\sigma[M]$ and a semiprime module. Suppose that M satisfies ACC on left annihilators. If $P_1, P_2, ..., P_n$ are the minimal prime in M submodules then the following conditions are equivalent:*

1. *M is a Goldie module.*

2. *$N_i = Ann_M(P_i)$ has finite uniform dimension for all $1 \leq i \leq n$.*

Proof. *1 ⇒ 2.* Since M is a Goldie module, M has finite uniform dimension. So, every N_i has finite uniform dimension.

2 ⇒ 1. By Proposition 3.2.9, M has finite uniform dimension. Thus M is a Goldie module. □

Corollary 4.2.4. *Let R be a semiprime ring such that R satisfies ACC on left annihilators. If $P_1, P_2, ..., P_n$ are the minimal prime ideals of R, then the following conditions are equivalent:*

1. *R is a left Goldie ring.*

2. *$Ann_R(P_i)$ has finite uniform dimension for each $1 \leq i \leq n$.*

Corollary 4.2.5. *Let R be a semiprime ring such that R satisfies ACC on left annihilators. If $P_1, P_2, ..., P_n$ are the minimal prime ideals of R, with $n > 1$ then the following conditions are equivalent:*

1. *R is a left Goldie ring.*

2. *P_i has finite uniform dimension for each $1 \leq i \leq n$.*

Proof. *1 ⇒ 2.* It is clear.

2 ⇒ 1. By Lemma 3.2.11 R has finite uniform dimension. Thus R is a left Goldie ring. □

Let us prove some lemmas in order to give an other decomposition of $\widehat{M}$ in terms of factor modules of M.

Lemma 4.2.6. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M satisfies ACC on left annihilators and $P_1, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$ for each $1 \leq i \leq n$ then P_i is a pseudocomplement of N_i which contains $\bigoplus_{j \neq i} N_j$ for all $1 \leq i \leq n$. Moreover $\bigoplus_{j \neq i} N_j \leq_e P_i$ for all $1 \leq i \leq n$.*

Proof. Fix $1 \leq i \leq n$. Let $L \leq M$ such that $P_i \leq L$ and $L \cap N_i = 0$. Since N_i is a fully invariant submodule of M then $N_{i_M} L \leq N_i \cap L = 0$. So $L \leq Ann_M(N_i) = Ann_M(Ann_M(P_i)) = P_i$. Thus $L = P_i$.

Now by Proposition 2.2.9 we have that $N_i = Ann_M(P_i) = \bigcap_{i \neq j} P_j$. Hence $N_j \leq P_i$ for all $j \neq i$. Then $\bigoplus_{j \neq i} N_j \leq P_i$ and by Lemma 4.2.1, $\bigoplus_{j \neq i} N_j \leq_e P_i$. □

Lemma 4.2.7. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M satisfies ACC on annihilators and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$, then $P_i + N_i \leq_e M$ for all $1 \leq i \leq n$.*

Proof. By Lemmas 3.2.7 and 4.2.6, N_i and P_i are pseudocomplements of each other for all $1 \leq i \leq n$, thus $N_i + P_i = N_i \oplus P_i \leq_e M$ for all $1 \leq i \leq n$. □

Lemma 4.2.8. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M satisfies ACC on left annihilators and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$, then*

$$P_i = M \cap \bigoplus_{j \neq i} E^{[M]}(N_j)$$

for all $1 \leq i \leq n$.

Proof. By Corollary 3.2.8 $\widehat{M} = E^{[M]}(N_1) \oplus ... \oplus E^{[M]}(N_n)$. By Lemma 4.2.7, $\widehat{M} = E^{[M]}(P_i) \oplus E^{[M]}(N_i)$ for each $1 \leq i \leq n$, moreover $E^{[M]}(P_i) = \bigoplus_{j \neq i} E^{[M]}(N_j)$ by Lemma 4.2.6.

We have that $N_i \cap (M \cap E^{[M]}(P_i)) = N_i \cap E^{[M]}(P_i) = 0$, since P_i is pseudocomplement of N_i then $P_i = M \cap E^{[M]}(P_i)$. Thus $P_i = M \cap \bigoplus\limits_{j \neq i} E^{[M]}(N_j)$ for all $1 \leq i \leq n$. □

Proposition 4.2.9. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M satisfies ACC on annihilators and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$ for $1 \leq i \leq n$, then the morphism*

$$\Psi : M \to M/P_1 \oplus M/P_2 \oplus ... \oplus M/P_n$$

given by $\Psi(m) = (m + P_1, m + P_{2,} ...., m + P_n)$ is a monomorphism and $Im\Psi \leq_e \bigoplus\limits_{i=1}^{n} M/P_i$.

Proof. By Corollary 2.1.38 we have that, $\bigcap\limits_{i=1}^{n} P_i = 0$. Thus Ψ is a monomorphism. Now let $0 \neq (m_1 + P_1, m_2 + P_{2,} ...., m_n + P_n) \in \bigoplus\limits_{i=1}^{n} M/P_i$. By Lemma 4.2.8, $\bigoplus\limits_{i=1}^{n} \left(P_i + \bigcap\limits_{i \neq j} P_j \right) \leq_e M^n$. So there exists $r \in R$ such that

$$0 \neq r(m_1, m_{2,} ...., m_n) \in \bigoplus_{i=1}^{n} \left(P_i + \bigcap_{i \neq j} P_j \right)$$

Hence $rm_i \in P_i + \bigcap\limits_{i \neq j} P_j$ for $1 \leq i \leq n$. Thus there exist $x_i \in P_i$ and $y_i \in \bigcap\limits_{i \neq j} P_j$ such that $rm_i = x_i + y_i$ for every $1 \leq i \leq n$. We claim that $r(m_1 + P_1, m_2 + P_{2,} ...., m_n + P_n) \in Im\Psi$. In fact let $m = y_1 + y_2 + ... + y_n$, then

$$\Psi(m) = (y_1 + y_2 + ... + y_n + P_1, ..., y_1 + y_2 + ... + y_n + P_n)$$

$$= (y_1 + P_1, ..., y_n + P_n) = (rm_1 - x_1 + P_1, ..., rm_n - x_n + P_n)$$

$$= r(m_1 + P_1, ..., m_n + P_n)$$

□

Corollary 4.2.10. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M satisfies ACC on left annihilators and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules, then:*

1. $\widehat{M} \cong E^{[M]}(M/P_1) \oplus E^{[M]}(M/P_2) \oplus ... \oplus E^{[M]}(M/P_n)$.
2. *M/P_i has finite uniform dimension for all $0 \leq i \leq n$, if M is a Goldie module.*

Proof. 1. By Proposition 4.2.9 Ψ is a monomorphism and $Im\Psi \leq_e \bigoplus_{i=1}^{n} M/P_i$. Thus we have the result.

2. Since M is a Goldie module, then M has finite uniform dimension. By (*1*) we have that M/P_i has finite uniform dimension for all $0 \leq i \leq n$. □

To prove that a module is a Goldie module, with the last corollary it is enough to prove that every factor module given by a minimal prime is a Goldie module.

Proposition 4.2.11. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M has finitely many minimal prime submodules $P_1, P_2, ..., P_n$. Then M is a Goldie module if and only if each M/P_i is a Goldie module.*

Proof. $\Rightarrow$. By Corollary 4.2.10.*2* if M is a Goldie module then each quotient M/P_i has finite uniform dimension. Notice that by Proposition 2.2.9

$$P_i \leq Ann_M(P_1 \cap ... \cap P_{i-1} \cap P_{i+1} \cap ... \cap P_n)$$

Since M has finite uniform dimension, there exists a uniform submodule U_i of $P_1 \cap ... \cap P_{i-1} \cap P_{i+1} \cap ... \cap P_n$. So $P_i \leq Ann_M(U_i)$. By Proposition 2.2.11 and Proposition 2.2.10, $P_i = Ann_M(U_i)$. Then, there exists a monomorphism $M/P_i \to {U_i}^X$ and since U_i is non M-singular, then M/P_i is non M-singular. Thus M/P_i is non (M/P_i)-singular. Since M/P_i is a prime module, by Theorem 4.1.20 M/P_i is a Goldie module.

$\Leftarrow$. By Corollary 2.1.38 there exists a monomorphism $M \to \bigoplus_{i=1}^{t} M/P_i$. Since each M/P_i has finite uniform dimension then M has finite uniform dimension.

Let $0 \neq N$ be a submodule of M. Since there exists a monomorphism $M \to \bigoplus M/P_i$ then there exists $1 \leq i \leq t$ and submodules $0 \neq K \leq M/P_i$ and $0 \neq N' \leq N$ such that $K \cong N'$. We have that M/P_i is a Goldie module, thus it has enough monoforms. Hence N' has a monoform submodule, that is M has enough monoforms, and so by Corollary 4.1.13, M is a Goldie module. □

Lemma 4.2.12. *Let M be projective in $\sigma[M]$. If M is a semiprime Goldie module then so is M^n for all $n > 0$.*

Proof. We have that $\sigma[M] = \sigma[M^n]$. Then M^n is projective in $\sigma[M^n]$. Since M has finite uniform dimension then so does M^n . By Theorem 4.1.10 M is essentially compressible, then M^n is essentially compressible by [27, Proposition 1.2]. Thus by Lemma 4.1.7 and Theorem 4.1.10 M^n is a semiprime Goldie module. □

Corollary 4.2.13. *Let R be a ring. If R is a semiprime left Goldie ring then $M_n(R)$ is a semiprime left Goldie ring for all $n > 0$.*

Proof. By Lemma 4.2.12, R^n is a semiprime Goldie module for all $n > 0$. By Theorem 4.1.25, $End_R(R^n) \cong M_n(R)^{op}$ is a semiprime right Goldie ring. □

Proposition 4.2.14. *Let M be projective in $\sigma[M]$. Suppose that M has nonzero socle. If M is a prime module and satisfies ACC on left annihilators then M is semisimple artinian and FI-simple.*

Proof. Let $0 \neq m \in M$. Since $Soc(M) \neq 0$ and M is a prime module then $0 \neq Soc(M)_M Rm \subseteq Soc(M) \cap Rm$. Hence $Soc(M) \leq_e M$. Let U_1 be a minimal submodule of M. By Corollary 2.1.46 U_1 is a direct summand, so $M = U_1 \oplus V_1$. If $0 \neq V_1$, since $Soc(M) \leq_e M$, then there exists a minimal submodule $U_2 \leq V_1$. Then $M = U_1 \oplus U_2 \oplus V_2$. If $V_2 \neq 0$ there exists a minimal submodule $U_3 \leq V_3$ such that $M = U_1 \oplus U_2 \oplus U_3 \oplus V_3$. Following in this way, we get an ascending chain

$$U_1 \leq U_1 \oplus U_2 \leq U_1 \oplus U_2 \oplus U_3 \leq ...$$

Notice that $U_1 \oplus ... \oplus U_i = Ker f$ where f is the endomorphism of M given by

$$M \to M/(U_1 \oplus ... \oplus U_i) \cong V_i \hookrightarrow M$$

Thus, the last chain is an ascending chain of annihilators, so it must stop in a finite step. Then M is semisimple artinian.

Suppose that $M = U_1 \oplus ... \oplus U_n$. If $Hom_R(U_i, U_j) = 0$ then $U_{iM}U_j = 0$, but M is prime. Thus $Hom_R(U_i, U_j) \neq 0$, so $U_i \cong U_j$ $1 \leq i, j \leq n$. □

Corollary 4.2.15. *Let M be projective in $\sigma[M]$. Suppose M has essential socle. If M is a semiprime Goldie module then M is semisimple artinian.*

Proof. By Proposition 4.2.9 there exists an essential monomorphism $\Psi : M \to M/P_1 \oplus \cdots \oplus M/P_n$ where P_i are the minimal prime in M submodules. Hence each M/P_i has nonzero socle and by Proposition 4.2.11 M/P_i is a Goldie module. Thus M/P_i is semisimple artinian and FI-simple for all $1 \leq i \leq n$ by Proposition 4.2.14. Then $M \cong M/P_1 \oplus \cdots \oplus M/P_n$ and hence semisimple artinian. □

Example 4.2.16. In $\mathbb{Z}-Mod$, a projective module M in $\sigma[M]$ is a semiprime Goldie module if and only if M is semisimple artinian or M is free of finite rank.

Proof. It is clear that a semisimple artinian module M is projective in $\sigma[M]$ and a semiprime Goldie module and, by Lemma 4.2.12, every free module of finite rank is a semiprime Goldie module. Now, suppose that ${}_{\mathbb{Z}}M$ is a semiprime Goldie module and projective in $\sigma[M]$. Recall that in $\mathbb{Z}$-Mod an indecomposable injective module is isomorphic to $\mathbb{Q}$ or $\mathbb{Z}_{p^\infty}$ for some prime p.

Let U be an uniform submodule of M. By definition $E^{[M]}(U) = tr^M(E^{[\mathbb{Z}]}(U))$. Suppose $E^{[\mathbb{Z}]}(U) \cong \mathbb{Q}$, then $E^{[M]}(U) \leq \mathbb{Q}$. Since $\mathbb{Q}$ is FI-simple then $E^{[M]}(U) = \mathbb{Q}$. This implies that $\mathbb{Q} \hookrightarrow \widehat{M} \in \sigma[M]$. Hence $\sigma[M] = \mathbb{Z} - Mod$. Since M is projective in $\sigma[M] = \mathbb{Z} - Mod$ then M is a free module and since M has finite uniform dimension then it has finite rank.

Let $\bigoplus_{i=1}^{n} U_i \leq_e M$ with U_i uniform. If one U_i is a torsion free group then M is free because above. So, we can suppose that every U_i is a torsion group. Then, M has essential socle. By Corollary 4.2.15 M is semisimple artinian. □

Proposition 4.2.17. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M is a Goldie module and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules, then $E^{[M]}(N_i) \cong E^{[M]}(M/P_i)$ where $N_i = Ann_M(P_i)$. Moreover, M/P_i contains an essential submodule isomorphic to N_i.*

Proof. By [7, Proposition 4.5], $Ass_M(M/P_i) = \{P_i\}$. By Corollary 4.2.10.*1* $\widehat{M} \cong E^{[M]}(M/P_1) \oplus E^{[M]}(M/P_2) \oplus ... \oplus E^{[M]}(M/P_n)$. Then, following the proof of Theorem 4.2.1 there exist indecomposable injective modules F_1,..,F_n in $\sigma[M]$ and $l_1, ..., l_n \in \mathbb{N}$ such that $\widehat{M} \cong F_1^{l_1} \oplus ... \oplus F_n^{l_n}$ and $E^{[M]}(M/P_i) \cong$

$F_i^{l_i}$. By Theorem 4.2.4 $\widehat{M} \cong E_1^{k_1} \oplus ... \oplus E_n^{k_n}$ with $E^{[M]}(N_i) \cong E_i^{k_i}$ and $k_1, ..., k_n \in \mathbb{N}$. Then by Krull-Remak-Schmidt-Azumaya Theorem, $E_i \cong F_i$ and $l_i = k_i$. Thus $E^{[M]}(M/P_i) \cong E^{[M]}(N_i)$.

Now, by Lemma 4.2.7, $N_i \oplus P_i \leq_e M$. Then there is an essential monomorphism $N_i \hookrightarrow M/P_i$. In fact, N_i and P_i are pseudocomplements one of each other. □

Lemma 4.2.18. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M is a Goldie Module and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules. If $N_i = Ann_M(P_i)$, then $Ann_M\left(E^{[M]}(N_i)\right) = P_i$ for all $1 \leq i \leq n$.*

Proof. Since N_i has finite uniform dimension

$$E^{[M]}(N_i) = E^{[M]}(U_{i_1}) \oplus ... \oplus E^{[M]}(U_{i_{k_i}})$$

where U_{i_j} is a uniform submodule of N_i. By Lemma 3.2.5, $Ann_M(U_{i_j}) = P_i$ for all $1 \leq j \leq k_i$. By Proposition 2.2.11 and Proposition 2.2.10 we have that $Ann_M(U_{i_j}) = Ann_M(E^M(U_{i_j}))$ for all $1 \leq j \leq k_i$. Finally, by [7, Proposition 1.3], $Ann_M(E^M(U_{i_1}) \oplus ... \oplus E^M(U_{i_{k_i}})) = \bigcap_{j=1}^{k_i} Ann_M(E^M(U_{i_j})) = P_i$. Thus $Ann_M\left(E^M(N_i)\right) = P_i$ for all $1 \leq i \leq n$. □

Proposition 4.2.19. *Let M be projective in $\sigma[M]$ and semiprime. Suppose that M is a Goldie Module and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules, then:*

1. *$Hom_R(E^{[M]}(M/P_i), E^{[M]}(M/P_j)) = 0$ if $i \neq j$.*

2. *If M is also a generator in $\sigma[M]$ then $E^{[M]}(M/P_i) = \widehat{M/P_i}$.*

Proof. *1.* It follows by the proof of Proposition 4.2.2 and Proposition 4.2.17.

2. Since $\sigma[M/P_i] \leq \sigma[M]$ for every $1 \leq i \leq n$, then $\widehat{M/P_i} \leq E^{[M]}(M/P_i)$. It is enough to show that $E^{[M]}(M/P_i) \in \sigma[M/P_i]$.

We claim that $P_{iM}E^{[M]}(M/P_i) = 0$. If $N_i = Ann_M(P_i)$, by Lemma 4.2.18 $Ann_M(E^{[M]}(N_i)) = P_i$. Now, by Proposition 4.2.17, $E^{[M]}(N_i) = E^{[M]}(M/P_i)$. Thus $P_{iM}E^{[M]}(M/P_i) = 0$.

By [6, Proposition 1.5] $E^{[M]}(M/P_i) \in \sigma[M/P_i]$. □

Notice that, *(2)* of last proposition is false if M is not a semiprime module.

Example 4.2.20. Let $M = \mathbb{Z}_4$, then M is a Goldie module but it is not semiprime. $2\mathbb{Z}_4$ is the only minimal prime in M and $\mathbb{Z}_4/2\mathbb{Z}_4 \cong \mathbb{Z}_2$. We have that $E^{[\mathbb{Z}_4]}(\mathbb{Z}_4/2\mathbb{Z}_4) = \mathbb{Z}_4$ but $E^{[\mathbb{Z}_2]}(\mathbb{Z}_4/2\mathbb{Z}_4) = \mathbb{Z}_2$.

Remark 4.2.21. Note that if M is a Goldie Module then, by 4.2.11, we have that M/P_i is a Goldie module for every minimal prime P_i submodule. If $S_i = End_R(M/P_i)$ and $T_i = End_R(\widehat{M/P_i})$, then by Corollary 4.1.26, T_i is a simple right artinian ring and is the classical right ring of quotients of S_i. On the other hand, by Proposition 4.2.19, $E^{[M]}(M/P_i) = \widehat{M/P_i}$. So $End_R(E^{[M]}(M/P_i)) \cong End_R\left(\widehat{M/P_i}\right)$. Thus $End_R(E^{[M]}(M/P_i))$ is a simple right artinian ring and is the classical right ring of quotients of S_i. Moreover by Proposition 4.2.19 and Proposition 4.2.17 we have that $End_R(E^{[M]}(M/P_i)) \cong End_R(E^{[M]}(N_i)) \cong End_R(\widehat{M/P_i})$.

Theorem 4.2.22. *Let M be progenerator in $\sigma[M]$ and semiprime. Suppose that M is a Goldie Module and $P_1, P_2, ..., P_n$ are the minimal prime in M submodules, then*

$$End_R(\widehat{M}) \cong End_R(E^{[M]}(M/P_1)) \times \cdots \times End_R(E^{[M]}(M/P_n))$$

where $End_R(E^{[M]}(M/P_i))$ is a simple right artinian ring and is the classical ring of quotients of $End_R(M/P_i)$.

Proof. By Corollary 4.2.10,

$$\widehat{M} \cong E^{[M]}(M/P_1) \oplus E^{[M]}(M/P_2) \oplus ... \oplus E^{[M]}(M/P_n)$$

Then

$$End_R(\widehat{M}) \cong End_R(E^{[M]}(M/P_1) \oplus E^{[M]}(M/P_2) \oplus ... \oplus E^{[M]}(M/P_n))$$

By Proposition 4.2.19, $Hom_R(E^{[M]}(M/P_i), E^{[M]}(M/P_j)) = 0$ if $i \neq j$, hence

$$End_R(\widehat{M}) \cong End_R(E^{[M]}(M/P_1)) \times \cdots \times End_R(E^{[M]}(M/P_n))$$

The rest follows by Remark 4.2.21. □

Corollary 4.2.23. *Let M be progenerator in $\sigma[M]$ and semiprime. Suppose that M is a Goldie Module and let $T = End_R(\widehat{M})$. Then there is a bijective correspondence between the set of minimal prime in M submodules and a complete set of isomorphism classes of simple T-Modules.*

Proof. By Theorem 4.2.22

$$End_R(\widehat{M}) \cong End_R(E^{[M]}(M/P_1)) \times \cdots \times End_R(E^{[M]}(M/P_n))$$

Each $End_R(E^{[M]}(M/P_i))$ is an homogeneous component of T. If H is a simple T-module, then $H \cong T/L$ where L is a maximal right ideal of T. Then $H \hookrightarrow End_R(E^{[M]}(M/P_i))$ for some $1 \leq i \leq n$. $\square$

Appendix A

Appendix

A.1 Quasi-Quantales

In this appendix, it will be given a generalization of a quantale. This new ordered structure will be helpful for studying the complete lattice of submodules of a module.

Definition A.1.1. Let A be a $\bigvee$-semilattice. A is a *quasi-quantale* if A has an associative product $A\times A \to A$ such that for all directed subsets $X, Y \subseteq A$ and $a \in A$:

$$(\bigvee X)a = \bigvee\{xa \mid x \in X\}$$

$$a(\bigvee Y) = \bigvee\{ay \mid y \in Y\}$$

We say A is a *left (resp. right, resp. two-sided) quasi-quantale* if there exists $e \in A$ such that $e(a) = a$ (resp. $(a)e = a$, resp. $e(a) = a = (a)e$) for all $a \in A$.

Example A.1.2.

1. It is clear that every quantale (Definition 2.1.15) is a quasi-quantale.

2. If A is a meet-continuous lattice, then A is a bilateral quasi-quantale with $\wedge\colon A \times A \to A$. See Definition 2.1.16.

3. Let R-pr be the big lattice of preradicals on R-Mod, where the order is given by: $r, s \in R-pr$ if and only if $r(M) \leq s(M)$ for all R-module M. If $r, s \in R-pr$ their product is defined as $(rs)(M) = r(s(M))$,

for every R-module M. With this product R-pr is a left quasi-quantale where e is the identity functor. See [21].

4. Let M be projective in $\sigma[M]$. Given $N, L \in \Lambda(M)$ we have the product

$$N_M L = \sum\{f(N)|f \in Hom_R(M, L)\}.$$

Then, by Lemma 2.1.13 and Lemma 2.1.19 $\Lambda(M)$ is a quasi-quantale. Note that in general, if $N \in \Lambda(M)$

$$M_M N < N < N_M M$$

So, $\Lambda(M)$ is not a right, left or bilateral quasi-quantale.

Proposition A.1.3. *Let A be a quasi-quantale and $x, y, z \in A$. Then, the following conditions hold.*

1. *If $x \leq y$, then $zx \leq zy$ and $xz \leq yz$.*

2. *Moreover, if $1a \leq a$ for all $a \in A$, then*

   (a) $xy \leq y \wedge x1$

   (b) $x0 = 0$

Proof. 1. Notice that $\{x, y\} \subseteq A$ is a directed subset, so $zy = z(x \vee y) = zx \vee zy$. Thus $zx \leq zy$. Analogously, $yz = (x \vee y)z = xz \vee yz$. So $xz \leq yz$.

2(a). We have that $\{y, 1\}$ is directed, so $x1 = x(y \vee 1) = xy \vee x1$. Then $xy \leq x1$. On the other hand, $y \geq 1y = (x \vee 1)y = xy \vee 1y = xy \vee y$, then $xy \leq y$. Thus $xy \leq x1 \wedge y$.

2(b). By (*1*), it follows that $x0 \leq x1 \wedge 0 = 0$. □

Proposition A.1.4. *Let A be a quasi-quantale. The following conditions hold.*

1. *If $x \leq y$ and $z \leq v$, then $xz \leq yv$.*

2. $xy \vee xz \leq x(y \vee z)$ *and* $yx \vee zx \leq (y \vee z)x$, *for all* $x, y, z \in A$.

3. *If $1a \leq a$ for all $a \in A$, then*

   (a) $xx := x^2 \leq x1$ *and* $x^2 \leq x$, *for every* $x \in A$.

(b) $(x \wedge y)^2 \leq xy$ *and* $(x1 \wedge y)^2 \leq xy$.

Proof. *1.* It follows from Proposition A.1.3.*2(a)*.

2. Since $y \leq y \vee z$, then $xy \leq x(x \vee z)$. Similarly, $z \leq y \vee z$ implies that $xz \leq x(y \vee z)$. Therefore, $xy \vee xz \leq x(y \vee z)$.

3(a). It follows from Proposition A.1.3.*2(a)*.

3(b). Since $x \wedge y \leq x$ and $x \wedge y \leq y$, by (*2*), we conclude that $(x \wedge y)^2 \leq xy$. On the other hand, since $x1 \wedge y \leq x1$ and $x1 \wedge y \leq y$. By hypothesis, (*1*) and the associativity of the product, we conclude that $(x1 \wedge y)^2 \leq (x1)y = x(1y) \leq xy$. □

Proposition A.1.5. *Let A be quasi-quantale. Consider the following statements.*

1. $xy = 0$ *if and only if* $x1 \wedge y = 0$, *for every* $x, y \in A$.

2. *If* $x^2 = 0$, *then* $x = 0$, *for every* $x \in A$.

Then, the condition 1 always implies 2. If in addition, A is a quasi-quantale which satisfies that $1a \leq a$ for all $a \in A$, the two conditions are equivalent.

Proof. First, we prove that *1* implies *2*. Let $x \in A$ be such that $x^2 = 0$. Then, by hypothesis it follows that $x = x \wedge x = 0$.

Now, suppose that $1a \leq a$ for all $a \in A$. It remains to prove that *2* implies *1*. Let $x, y \in A$ be such that $xy = 0$. Then, by Proposition A.1.4.*3(b)*, we have that $(x1 \wedge y)^2 \leq xy = 0$, so $(x1 \wedge y)^2 = 0$. Hence, $x1 \wedge y = 0$. Conversely, suppose that $x1 \wedge y = 0$. Then, it is immediately that $xy \leq x1 \wedge y = 0$. □

Next two lemmas show when a quasi-quantale is a known lattice structure like a quantale or a meet-continuous lattice.

Lemma A.1.6. *Let A be a quasi-quantale which satisfies the following identities:*

$$a(b \vee c) = ab \vee ac$$

$$(b \vee c)a = ba \vee ca$$

for all $a, b, c \in A$. Then A is a quantale.

Proof. Let $X \subseteq A$. Define $Y = \{x_1 \vee ... \vee x_n \mid x_i \in X\}$. Then $X \subseteq Y$, so $\bigvee X \leq \bigvee Y$ and Y is a direct subset of A. Since A is quasi-quantale

$$a(\bigvee Y) = \bigvee\{ay \mid y \in Y\}.$$

Every $y \in Y$ is of the form $y = x_1 \vee ... \vee x_n$ with $x_i \in X$. Since $x_1 \vee ... \vee x_n \leq \bigvee X$ then

$$\bigvee\{ay \mid y \in Y\} \leq a(\bigvee X) \leq a(\bigvee Y),$$

so

$$a(\bigvee X) = \bigvee\{ay \mid y \in Y\}.$$

We have that $X \subseteq Y$, whence $\bigvee\{ax \mid x \in X\} \leq \bigvee\{ay \mid y \in Y\}$. On the other hand, by hypothesis

$$ay = a(x_1 \vee ... \vee x_n) = ax_1 \vee ... \vee ax_n \leq \bigvee\{ax \mid x \in X\},$$

thus $\bigvee\{ay \mid y \in Y\} \leq \bigvee\{ax \mid x \in X\}$. Hence

$$a(\bigvee X) = \bigvee\{ax \mid x \in X\}$$

□

Lemma A.1.7. *Let $(A, \leq, \vee, 1)$ be a complete lattice. Then, A is meet-continuous if and only if A is a quasi-quantale such that*

1. *The binary operation in A is commutative.*
2. *A has an identity , $e = 1$.*

Proof. First, suppose that A is a meet-continuous complete lattice. It is clear that A with the binary operation $\wedge$ is a quasi-quantale which satisfies *1* and *2*.

Conversely, since A is commutative, it follows that $1a = a = a1$. This implies that $ab \leq a \wedge b$. Thus $ab = a \wedge b$. Therefore, A is meet-continuous. □

For studying lattice structures, there is a very appropriated tool.

Definition A.1.8. An *inflator* on a complete lattice A is a function $d \colon A \to A$ such that $x \leq d(x)$ and $x \leq y \Rightarrow d(x) \leq d(y)$.

Let us denote by $I(A)$ the set of all inflators on A. In fact, $I(A)$ is a complete lattice with order given by

$$d \leq d' \Leftrightarrow d(a) \leq d'(a)$$

for all $a \in A$. For a family of inflators $\{d_i\}_I$ its supremum and infimum are described as the inflators

$$\left(\bigvee_I d_i\right)(a) = \bigvee_I (d_i(a))$$

$$\left(\bigwedge_I d_i\right)(a) = \bigwedge_I (d_i(a))$$

for all $a \in A$.

The least and greatest elements of $I(A)$ are the identity $d_{\underline{0}}$ of A and the constant function $\bar{d}(a) = \bar{1}$ for all $a \in A$, respectively.

From the definition of inflator, the composition of any two inflators is again an inflator. Moreover, with the next Lemma, $I(A)$ is a quasi-quantale

Lemma A.1.9. *If A is an idiom and d, d', k are inflators over A, then:*

1. *If $d \leq d'$, then $kd \leq kd'$ and $dk \leq d'k$.*
2. *$kd' \vee kd \leq k(d' \vee d)$ and $k(d' \wedge d) \leq kd' \wedge kd$.*
3. *Moreover, if $\mathcal{D} \subseteq I(A)$ is non empty, then:*
   (a) $(\bigvee \mathcal{D})k = \bigvee \{dk \mid d \in \mathcal{D}\}$,
   (b) $(\bigwedge \mathcal{D})k = \bigwedge \{dk \mid d \in \mathcal{D}\}$.
   If $\mathcal{D}$ is directed then
   (c) $k(\bigvee \mathcal{D}) = \bigvee \{kd \mid d \in \mathcal{D}\}$

In the inflators theory, there are remarkable kinds of inflators

Definition A.1.10. A *pre-nucleus* d on A is an inflator such that $d(x \wedge y) = d(x) \wedge d(y)$. A *Closure Operator* is an idempotent inflator. If d is a closure operator and a pre-nucleus then d is called *Nucleus*.

Definition A.1.11. Let A be a quasi-quantale. Consider $s\colon A \to A$ a inflator on A. We say that:

1. s is *contextual stable* if $s(a)x \leq s(ax)$ and $y(s(a)) \leq s(ya)$ for all $a, x, y \in A$.

2. s is *pre-multiplicative* if $s(a) \wedge b \leq s(ab)$ for all $a, b \in A$.

3. s is *multiplicative* if $s(a) \wedge s(b) = s(ab)$ for all $a, b \in A$.

4. s is a *contextual pre-nucleus* if $s(a)s(b) \leq s(ab)$ for all $a, b \in A$.

5. s is a *contextual nucleus* if $s^2 = s$ and s is a contextual pre-nucleus.

Proposition A.1.12. *Let A be a quasi-quantale and j be a contextual nucleus. Then $(A_j, \cdot)$ is a quasi-quantale. If A is a left quasi-quantale with identity e then A_j is a left quasi-quantale with identity $j(e)$.*

Proof. Let $a, b, c \in A_j$. Consider the following inequalities,

$$abc \leq j(ab)c \leq j(j(ab)c) = j(ab) \cdot c = (a \cdot b) \cdot c \tag{A.1}$$

$$j(ab)c = j(ab)j(c) \leq j(abc) \tag{A.2}$$

Using (2), we have

$$(a \cdot b) \cdot c = j(j(ab)c) \leq j^2(abc) = j(abc). \tag{A.3}$$

By (1), $j(abc) \leq (a \cdot b) \cdot c$, and by (3), we have the equality

$$(a \cdot b) \cdot c \leq j(abc) \leq (a \cdot b) \cdot c$$

Analogously, $a \cdot (b \cdot c) = j(abc)$. Then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Now, let $X \subseteq A_j$ be a directed subset and $a \in A_j$. Then,

$$a \cdot (\bigvee^{j} X) = j(a \bigvee^{j} X) = j(aj(\bigvee X)) \geq a(\bigvee X) = \bigvee\{ax|x \in X\}.$$

Notice that $j(ax) \leq j(\bigvee\{ax \mid x \in X\})$ for all $x \in X$, so

$$j(a \cdot (\bigvee^{j} X)) \geq j(\bigvee\{ax|x \in X\}) \geq \bigvee\{j(ax)|x \in X\}.$$

Since j is a nucleus,

$$a \cdot (\bigvee^{j} X) = j(a \cdot (\bigvee^{j} X)) \geq j(\bigvee \{j(ax) \mid x \in X\} = \bigvee^{j} \{a \cdot x \mid x \in X\}.$$

On the other hand,

$$\bigvee^{j} \{a \cdot x \mid x \in X\} = j(\bigvee \{j(ax) \mid x \in X\}) \geq j(\bigvee \{ax \mid x \in X\})$$

$$= j(a \bigvee X) \geq j(a)j(\bigvee X) = a \cdot \bigvee^{j} X$$

Thus

$$a \cdot \bigvee^{j} X \geq \bigvee^{j} \{ax | x \in X\} \geq a \cdot \bigvee^{j} X$$

Hence $(A_j, \cdot)$ is a quasi-quantale.

Now, suppose that A is a left quasi-quantale. Let $a \in A_j$, then

$$a = ea \leq j(e)a \leq j(j(e)a) = j(j(e)j(a)) \leq j(ea) = j(a) = a.$$

So $a = j(j(e)a) = j(e) \cdot a$. Hence A_j is a left quasi-quantale. $\square$

Proposition A.1.13. *Let A be a quasi-quantale. Then, for each multiplicative nucleus d on A, the set A_d is a meet-continuous lattice.*

Proof. Let $X \subseteq A_d$ be a directed subset and $a \in A_d$. Then,

$$d((\bigvee X) \wedge a) = d((\bigvee X)a) = d(\bigvee \{xa | x \in X\}) \leq d(\bigvee \{d(xa) | x \in X\})$$

$$= d(\bigvee \{d(x) \wedge d(a) | x \in X\}) = d(\bigvee \{x \wedge a | x \in X\}) \leq d((\bigvee X) \wedge a).$$

Therefore, $d((\bigvee X) \wedge a) = d(\bigvee \{x \wedge a | x \in X\})$. Thus

$$(\bigvee^{d} X) \wedge a = d(\bigvee X) \wedge d(a) = d((\bigvee X) \wedge a)$$

$$= d(\bigvee \{x \wedge a | x \in X\}) = \bigvee^{d} \{x \wedge a | x \in X\}.$$

$\square$

Corollary A.1.14. *Let A be a quasi-quantale. Suppose that for any $X \subseteq A$ and $a \in A$ is satisfied*

$$(\bigvee X)a = \bigvee\{xa \mid x \in X\}.$$

Then, for each multiplicative nucleus d on A, the set A_d is a frame.

Definition A.1.15. Let A be a quasi-quantale. An element $1 \neq p \in A$ is *prime* if whenever $ab \leq p$ then $a \leq p$ or $b \leq p$.

Definition A.1.16. Let A be a quasi-quantale and B a sub $\bigvee$-semilattice. We say that B is a *subquasi-quantale* of A if

$$(\bigvee X)a = \bigvee\{xa \mid x \in X\}$$

and

$$a(\bigvee Y) = \bigvee\{ay \mid y \in Y\},$$

for all directed subsets $X, Y \subseteq B$ and $a \in B$.

Definition A.1.17. Let A be a quasi-quantale and B a subquasi-quantale of A. An element $1 \neq p \in A$ is a *prime element relative to B* if whenever $ab \leq p$ with $a, b \in B$ then $a \leq p$ or $b \leq p$.

It is clear that a prime element in A is a prime element relative to A. The spectrum relative to B of A is defined as

$$Spec_B(A) = \{p \in A \mid p \textit{ is prime relative to } B\}$$

If $B = A$, then we just write $Spec(A)$.

Lemma A.1.18. *Let B be a subquasi-quantale of a quasi-quantale A. Suppose that $0, 1 \in B$ and $1b, b1 \leq b$ for all $b \in B$, then:*

1. *$ab \leq a \wedge b$ for all $a, b \in B$.*
2. *Let $p \in A$ a prime element relative to B. If $a, b \in B$, then $ab \leq p$ if and only if $a \leq p$ or $b \leq p$.*

Proof. (1). It follows from Proposition A.1.3.*2.*(a) that $ab \leq a \wedge 1b$. By hypothesis $1b \leq b$, so $ab \leq a \wedge b$.

(2). Let $a, b \in B$ and $p \in A$ a prime element relative to B. First, if $ab \leq p$, then, by definition, it follows that $a \leq p$ or $b \leq p$.

Conversely, suppose that $a \leq p$. By (1), $ab \leq a \wedge b \leq a$, then $ab \leq p$. Analogously if $b \leq p$. □

Proposition A.1.19. *Let B be a subquasi-quantale of a quasi-quantale A. Then $Spec_B(A)$ is a topological space, where the closed subsets are subsets given by*

$$\mathcal{V}(b) = \{p \in Spec_B(A) \mid b \leq p\},$$

with $b \in B$.

In dual form, the open subsets are of the form

$$\mathcal{U}(b) = \{p \in Spec_B(A) \mid b \nleq p\},$$

with $b \in B$.

Proof. It clear that $\mathcal{U}(0) = \emptyset$ and $\mathcal{U}(1) = Spec_B(A)$. Let $\{b_i\}_I$ be a family of elements in B. Then

$$\mathcal{U}(\bigvee_I b_i) = \{p \in Spec_B(A) \mid \bigvee_I b_i \nleq p\} = \bigcup_I \{p \in Spec_B(A) \mid b_i \nleq p\}$$

$$= \bigcup_I \mathcal{U}(b_i)$$

Now, let $a, b \in B$. Then, by Lemma A.1.18.(2),

$$\mathcal{U}(ab) = \{p \in Spec_B(A) \mid ab \nleq p\} = \{p \in Spec_B(A) \mid a \nleq p \text{ and } b \nleq p\}$$

$$= (\{p \in Spec_B(A) \mid a \nleq p\}) \cap (\{p' \in Spec_B(A) \mid b \nleq p'\}) = \mathcal{U}(a) \cap \mathcal{U}(b).$$

□

Remark A.1.20. Let $\mathcal{O}(Spec_B(A))$ be the frame of open subsets of $Spec_B(A)$. We have an adjunction of $\bigvee$-morphisms

$$\begin{tikzcd} B \arrow[r, bend left, "\mathcal{U}"] & \mathcal{O}(Spec_B(A)) \arrow[l, bend left, "\mathcal{U}_*"] \end{tikzcd}$$

where $\mathcal{U}_*$ is defined as

$$\mathcal{U}_*(W) = \bigvee\{b \in B \mid \mathcal{U}(b) \subseteq W\}$$

The composition $\mu = \mathcal{U}_* \circ \mathcal{U}$ is a closure operator in B.

Proposition A.1.21. *Let $b \in B$. Then $\mu(b)$ is the largest element in B such that $\mu(b) \leq \bigwedge\{p \in Spec_B(A) \mid p \in \mathcal{V}(b)\}$.*

Proof. By definition,

$$\mu(b) = \mathcal{U}_*(\mathcal{U}(b)) = \bigvee\{c \in B \mid \mathcal{U}(c) \subseteq \mathcal{U}(b)\}$$

$$= \bigvee\{c \in B \mid \mathcal{V}(b) \subseteq \mathcal{V}(c)\} \leq \bigwedge\{p \in Spec_B(A) \mid p \in \mathcal{V}(b)\}.$$

Let $x \in B$ such that $x \leq p$ for all $p \in \mathcal{V}(b)$, then $\mathcal{V}(b) \subseteq \mathcal{V}(x)$. Thus, $x \leq \bigvee\{c \in A \mid \mathcal{V}(b) \subseteq \mathcal{V}(c)\}$, whence $x \leq \mu(b)$. □

Theorem A.1.22. *The closure operator in $\mu\colon B \to B$ is a multiplicative pre-nucleus.*

Proof. Let $a, b \in B$. By Lemma A.1.18.(1), $ab \leq a \wedge b$. Thus $\mu(ab) \leq \mu(a) \wedge \mu(b)$.

By Proposition A.1.21,

$$\mu(a) \wedge \mu(b) \leq (\bigwedge\{q \in Spec_B(A) | q \in \mathcal{V}(a)\}) \wedge (\bigwedge\{q' \in Spec_B(A) \mid q' \in \mathcal{V}(b)\}).$$

Let $p \in Spec_B(A)$ such that $ab \leq p$, then $a \leq p$ or $b \leq p$. If $a \leq p$ then,

$$\bigwedge\{q \in Spec_B(A) \mid q \in \mathcal{V}(a)\}\} \leq p.$$

Thus, $\mu(a) \wedge \mu(b) \leq p$. Analogously, if $b \leq p$ then $\mu(a) \wedge \mu(b) \leq p$. Hence

$$\mu(a) \wedge \mu(b) \leq \bigwedge\{p \in Spec_B(A) \mid p \in \mathcal{V}(ab)\}$$

By Proposition A.1.21, $\mu(ab)$ is the largest element in B less or equal than $\bigwedge\{p \in Spec_B(A) \mid p \in \mathcal{V}(ab)\}$, therefore $\mu(a) \wedge \mu(b) \leq \mu(ab)$. Thus μ is multiplicative.

Now, since $ab \leq a \wedge b$ then $\mu(a) \wedge \mu(b) = \mu(ab) \leq \mu(a \wedge b)$. The other inequality always holds. Thus μ is a pre-nucleus. □

Corollary A.1.23. *A_μ is an meet-continuous lattice.*

Proof. It follows by Proposition A.1.13. □

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