SEMINAR ON "TILTING MODULES AND TILTING TORSION THEORIES" WRITTEN BY R. COLPI AND J. TRLIFAJ

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Abstract. These notes were made during a graduate seminar at Benemérita Universidad Autónoma de Puebla in Spring-2020. In the seminar, we studied the paper [4].

1. Preliminaries

It is assumed that the reader is familiar with the left and right derived functors. The functor Ext will play a central roll in this notes. The general background on derived functors can be found in [13]. For convenience of the reader we will mention some results which will be used along the manuscript.

Proposition 1.1. *(1) If $\{A_k\}_{k\in K}$ is a family of modules, then there are natural isomorphisms, for all $n > 0$,*

$$\mathrm{Ext}^n_R\left(\bigoplus_{k\in K} A_k, B\right) \cong \prod_{k\in K} \mathrm{Ext}^n_R(A_k, B).$$

(2) If $\{B_k\}_{k\in K}$ is a family of modules, then there are natural isomorphisms, for all $n > 0$,

$$\mathrm{Ext}^n_R\left(A, \prod_{k\in K} B_k\right) \cong \prod_{k\in K} \mathrm{Ext}^n_R(A, B_k).$$

Proof. [13, Proposition 7.21 and 7.22]. □

Proposition 1.2. *(1) A left R-module P is projective if and only if $\mathrm{Ext}^n_R(P, B) = 0$ for every R-module B.*
(2) A left R-module E is injective if and only if $\mathrm{Ext}^n_R(A, E) = 0$ for every R-module A.

Proof. *(1)* [13, Corollary 6.58 and Corollary 7.25].
(2) [13, Corollary 6.41 and Corollary 7.25]. □

Definition 1.3. Let A be a left R-module. The *projective dimension of* A is a less or equal than n ($pd(A) \leq n$), if there is a finite projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0.$$

If no such finite resolution exists, then $pd(A) = \infty$; otherwise, $pd(A) = n$ if n is the length of a shortest projective resolution of A.

Proposition 1.4. *The following are equivalent for a left R-module A.*
(a) $pd(A) \leq n$
(b) $\mathrm{Ext}^k_R(A, B) = 0$ *for all left R-modules B and all $k \geq n + 1$.*

Proof. [13, Proposition 8.6]. □

Definition 1.5. A ring R is said to be *left hereditary* if every left ideal is projective.

Proposition 1.6. *The following conditions are equivalent for a ring R:*

(a) *R is left hereditary;*
(b) *Submodules of projective left R-modules are projective;*
(c) *Factor modules of injective left R-modules are injective.*

Corollary 1.7. *Let R be a left hereditary ring. Then $pd(A) \leq 1$ for all left R-module A.*

Given two modules A and C, an extension of A by C is an exact sequence $0 \to A \to B \to C \to 0$. The next Proposition shows that $\mathrm{Ext}^1(C, A)$ detects the nontrivial extensions of A by C.

Proposition 1.8. *Every extension $0 \to A \to B \to C \to 0$ splits if and only if $\mathrm{Ext}^1(C, A) = 0$.*

Proof. [13, Proposition 7.24 and Theorem 7.31]. □

I want to show the general idea of how from an extension of A by C we get an element in $\mathrm{Ext}^1(C, A)$ and vice-versa, all the details can be found in [13, Ch. 7, Sec. 2]. To get this we will need the constructions of pullback and pushout in modules. Let us start with $[\alpha] \in \mathrm{Ext}^1(C, A)$. Taking an injective resolution

$$\mathsf{E} \qquad 0 \longrightarrow A \xrightarrow{\eta} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \longrightarrow$$

of A and applying the functor $\mathrm{Hom}(C, \_)$ to the reduced resolution E_A we get the complex:

$$\mathrm{Hom}(C, \mathsf{E}_A) \qquad 0 \longrightarrow \mathrm{Hom}(C, E_0) \xrightarrow{(C,d_0)} \mathrm{Hom}(C, E_{-1}) \xrightarrow{(C,d_{-1})}$$

Then $\mathrm{Ext}^1(C, A) := H^1(\mathrm{Hom}(C, \mathsf{E}_A)) = \mathrm{Ker}(C, d_{-1}) / \mathrm{Im}(C, d_0)$. This implies that $[\alpha] \in \mathrm{Ext}^1(C, A)$ is represented by a morphism $\alpha : C \to E_{-1}$ such that $d_{-1}\alpha = 0$. Hence $\alpha(C) \subseteq \mathrm{Ker}\, d_{-1} = \mathrm{Im}\, d_0$. Therefore, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & M & \dashrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & A & \xrightarrow[\eta]{} & E_0 & \xrightarrow[d_0]{} & \mathrm{Im}\, d_0 & \longrightarrow & 0
\end{array}$$

where M is the pullback of the angle given by α and d_0. Thus, we have an extension of A by C.

Remark 1.9. The construction that we just made, can be done using a projective resolution of C. In this case, $\alpha : P_1 \to A$ and the extension is given by a pushout (see [13, Theorem 7.30]).

Conversely, suppose that we have an extension $0 \to A \to B \to C \to 0$ of A by C. Consider the injective resolution E of A. Then we have a morphism of complexes over the identity 1_A:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\
0 & \longrightarrow & A & \xrightarrow[\eta]{} & E_0 & \xrightarrow[d_0]{} & E_{-1} & \xrightarrow[d_{-1}]{} & E_{-2}
\end{array}$$

Hence $d_{-1}\alpha_1 = \alpha_2 0 = 0$. This implies that $\alpha_1 \in \operatorname{Ker}(C, d_{-1})$. Thus, $[\alpha_1] \in \operatorname{Ext}^1(C, A)$.

Lemma 1.10. *Let* $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\rho} C \longrightarrow 0$ *be an exact sequence and let* M *be a module. Then the connection morphism* $\partial : \operatorname{Hom}(M, C) \to \operatorname{Ext}^1(M, A)$ *is given by taking the pullback along* ρ. *That is, given* $f \in \operatorname{Hom}(M, A)$, $\partial(f)$ *corresponds to the extension:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{j} & L & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle -f} & & \\
0 & \longrightarrow & A & \xrightarrow[i]{} & B & \xrightarrow[\rho]{} & C & \longrightarrow & 0
\end{array}$$

Proof. We have to recall how the connection morphism was defined. Let us take injective resolutions

$$\mathsf{E}' \qquad 0 \longrightarrow A \xrightarrow{\eta'} E'_0 \xrightarrow{d'_0} E'_{-1} \xrightarrow{d'_{-1}} E'_{-2} \longrightarrow$$

$$\mathsf{E}'' \qquad 0 \longrightarrow C \xrightarrow{\eta''} E''_0 \xrightarrow{d''_0} E''_{-1} \xrightarrow{d''_{-1}} E''_{-2} \longrightarrow$$

of A by C respectively. Using the dual version of the horseshoe lemma, we have an injective resolution

$$\mathsf{E} \qquad 0 \longrightarrow B \xrightarrow{\eta} E_0 \xrightarrow{d_0} E_{-1} \xrightarrow{d_{-1}} E_{-2} \longrightarrow$$

of B such that $E_j = E'_j \oplus E''_j$. Therefore, there is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\rho} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta'} & \swarrow{\scriptstyle \sigma_0} & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta''} & & \\
0 & \longrightarrow & E'_0 & \xrightarrow{\zeta_0} & E_0 & \xrightarrow{\xi_0} & E''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d'_0} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d''_0} & & \\
0 & \longrightarrow & E'_{-1} & \xrightarrow{\zeta_{-1}} & E_{-1} & \xrightarrow{\xi_{-1}} & E''_{-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d'_{-1}} & & \downarrow{\scriptstyle d_{-1}} & & \downarrow{\scriptstyle d''_{-1}} & & \\
0 & \longrightarrow & E'_{-2} & \xrightarrow{\zeta_{-2}} & E_{-2} & \xrightarrow{\xi_{-2}} & E''_{-2} & \longrightarrow & 0
\end{array}$$

Applying the functor to the reduced resolutions, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(M,E_0') & \xrightarrow{(M,\zeta_0)} & \mathrm{Hom}(M,E_0) & \xrightarrow{(M,\xi_0)} & \mathrm{Hom}(M,E_0'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (M,d_0')} & & \downarrow{\scriptstyle (M,d_0)} & & \downarrow{\scriptstyle (M,d_0'')} & & \\
0 & \longrightarrow & \mathrm{Hom}(M,E_{-1}') & \xrightarrow{(M,\zeta_{-1})} & \mathrm{Hom}(M,E_{-1}) & \xrightarrow{(M,\xi_{-1})} & \mathrm{Hom}(M,E_{-1}'') & \longrightarrow & 0
\end{array}$$

Moreover, $\mathrm{Ker}(M,d_0'') \cong \mathrm{Hom}(M,C)$, where the isomorphism is given by $\eta'' \circ \_$. Now, $\eta : B \to E_0$ is defined as $\eta(b) = (\sigma_0(b), \eta''\rho(b))$ where $\sigma_0 : B \to E_0'$ is such that $\sigma_0 i = \eta'$. Analogously, $d_0 : E_0 \to E_{-1}$ is defined as $d_0(x) = (\sigma_1(\overline{x}), d_0''\xi_0(x))$ where $\sigma_1 : \mathrm{Coker}\,\eta \to E_{-1}'$. Then $\partial(f) = [\alpha] \in \mathrm{Ext}^1(M,A)$ is given by the morphism $\alpha \in \mathrm{Hom}(M, E_{-1}')$ such that $\alpha = \widehat{\zeta_{-1}} d_0 \overline{\xi_0} \eta'' f$ where $\overline{\xi_0} : E_0'' \to E_0$ is defined as $\xi_0(x) = (0,x)$ and $\widehat{\zeta_{-1}} : E_{-1} \to E_{-1}'$ is defined as $\widehat{\zeta_{-1}}(x,y) = x$. Set $\beta = \widehat{\zeta_{-1}} d_0 \overline{\xi_0} \eta''$. Note that

$$\begin{aligned}
\beta\rho(b) &= \widehat{\zeta_{-1}} d_0(0, \eta''\rho(b)) = \widehat{\zeta_{-1}}(\sigma_1\overline{(0,\eta''\rho(b))}, d_0''\eta''\rho(b)) \\
&= \widehat{\zeta_{-1}}(\sigma_1\overline{(0,\eta''\rho(b))}, 0) = \sigma_1\overline{(0,\eta''\rho(b))}.
\end{aligned}$$

On the other hand, $d_0'\sigma_0(b) = \zeta_{-1}\sigma_1\overline{(\zeta_0\sigma_0(b))} = \sigma_1\overline{(\sigma_0(b),0)}$. Thus, we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{j} & L & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \sigma_0\gamma} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & A & \xrightarrow{\eta'} & E_0' & \xrightarrow{d_0'} & \mathrm{Im}\, d_0' & \longrightarrow & 0.
\end{array}$$

Let us see that this diagram is commutative. Let $a \in A$. Then $\sigma_0\gamma j(a) = \sigma_0\gamma(i(a),0) = \sigma_0(i(a)) = \sigma_0(i(a)) = \eta'(a) = \eta'(1(a))$. On the other hand, $d_0'\sigma_0\gamma(b,m) = d_0'\sigma_0(b) = \sigma_1\overline{(\sigma_0(b),0)} = -\sigma_1\overline{(0,\eta''\rho(b))} = -\beta\rho(b)$. Since $(b,m) \in L$, $\rho(b) = -f(m)$. Then $d_0'\sigma_0\gamma(b,m) = -\beta\rho(b) = \beta f(m) = \alpha\pi(b,m)$. This implies that $\partial(f) = [\alpha] \in \mathrm{Ext}^1(M,A)$ is the element which corresponds to the extension

$$0 \longrightarrow A \xrightarrow{j} L \xrightarrow{\pi} M \longrightarrow 0$$

□

Definition 1.11. Let M be an R-module. The *Ext-orthogonal class of* M is given by

$$M^{\perp} = \{{}_R N \mid \mathrm{Ext}_R^1(M,N) = 0\}.$$

Definition 1.12. Given two R-modules M and N, it is said that N is M-generated if there exists an epimorphism $\rho : M^{(X)} \to N$ for some set X. The class of modules M-generated is denoted by $\mathrm{Gen}(M)$.

It is not difficult to see that the class $M^{\perp}$ is closed under extensions and contains all the injective R-modules by Proposition 1.2, and the class $\mathrm{Gen}(M)$ is closed under direct sums and epimorphisms, for all modules M.

Definition 1.13. A module M is *finendo* if M is finitely generated as module over its endomorphism ring.

Lemma 1.14 (Lemma 1.5, [3]). *Let M be a module. Then, $M^X \in \mathrm{Gen}(M)$ for every set X if and only if M is finendo.*

Proof. $\Rightarrow$ Suppose that $M^M \in \mathrm{Gen}({}_RM)$. Then, there exists an R-epimorphism $\rho : M^{(I)} \to M^M$ for some set I. Let $(x_m)_{m\in M}$ be the element in M^M such that $x_m = m$. Hence, there exists $(y_x)_{i\in I} \in M^{(I)}$ such that $\rho((y_i)_{i\in I}) = (x_m)_{m\in M}$. This implies that there there is a finite subset $F \subseteq I$ and homomorphisms $\rho_i : M \to M^M$ such that $(x_m)_{m\in M} = \rho((y_i)_{i\in I}) = \sum_{i\in F} \rho_i(y_i)$. For each $m \in M$, let π_m denote the canonical projection $\pi_m : M^M \to M$ and set $f_i^m = \pi_m\rho_i \in \mathrm{End}_R(M)$. Therefore,

$$m = \pi_m((x_m)_{m\in M}) = \pi_m\left(\sum_{i\in F} \rho_i(y_i)\right) = \sum_{i\in F} (\pi_m\rho_i(y_i)) = \sum_{i\in F} f_i^m(y_i),$$

for each $m \in M$. Thus, M as module over $\mathrm{End}_R(M)$ is generated by $\{y_i \mid i \in F\}$.

$\Leftarrow$ Let $S = \mathrm{End}_R(M)$ and suppose ${}_SM$ is generated by $\{y_1, ..., y_n\}$. Let X be any set and $(m_x)_{x\in X} \in M^X$. For each $x \in X$ there exist $f_1^x, ..., f_n^x \in S$ such that $m_x = \sum_{i=1}^n f_i^x(y_i)$. Let $\phi_x : M^n \to M$ be the homomorphism given by $\phi(m_1, ..., m_n) = \sum_{i=1}^n f_i^x(m_i)$ and let $\phi : M^n \to M^X$ be the homomorphism given by $\phi(m_1, ..., m_n) = (\phi_x(m_1, ..., m_n))_{x\in X}$. Consider the element $(y_1, ..., y_n) \in M^n$. Then,

$$\phi(y_1, ..., y_n) = (\phi_x(y_1, ..., y_n))_{x\in X} = \left(\sum_{i=1}^n f_i^x(y_i)\right)_{x\in X} = (m_x)_{x\in X}.$$

This implies that $(m_x)_{x\in X} \in tr^M(M^X)$. Thus, M^X is M-generated. □

Lemma 1.15. *Let T be an R-module. If $\mathrm{Gen}(T) = T^\perp$, then $\mathrm{Gen}(T) = \mathrm{Pres}(T)$.*

Proof. Let $M \in \mathrm{Gen}(T)$ and $X = \mathrm{Hom}_R(T, M)$. Then, there is an exact sequence $0 \to K \to T^{(X)} \overset{\rho}{\to} M \to 0$. Applying the functor $\mathrm{Hom}_R(T, \_)$, we get

$$\longrightarrow \mathrm{Hom}_R(T, T^{(X)}) \xrightarrow{\rho_*} \mathrm{Hom}_R(T, M) \longrightarrow \mathrm{Ext}^1(T, K) \longrightarrow 0$$

Note that $\mathrm{Ext}^1(T, T^{(X)}) = 0$, by hypothesis. We claim that ρ_* is surjective. For, consider $h \in \mathrm{Hom}_R(T, M)$. Let $\eta_h : T \to T^{(X)}$ be the canonical inclusion. Then, $\rho_*(\eta_h)(t) = \rho\eta_h(t) = h(t)$. Thus ρ_* is surjective and hence $Ext^1(T, K) = 0$. This implies that $K \in T^\perp = \mathrm{Gen}(T)$. So, $\mathrm{Gen}(T) \subseteq \mathrm{Pres}(T)$. We always have that $\mathrm{Pres}(T) \subseteq \mathrm{Gen}(T)$. □

Lemma 1.16. *Let M and N be R-modules such that $N \in \mathrm{Pres}(M)$ and $\mathrm{Gen}(M) \subseteq N^\perp$. Then, $N \in \mathrm{Add}(M)$.*

Proof. By hypothesis, there is an exact sequence $0 \to K \to M^{(X)} \to N \to 0$ with $K \in \mathrm{Gen}(M)$. Since $\mathrm{Gen}(M) \subseteq N^\perp$, $\mathrm{Ext}^1(N, K) = 0$. This implies that the sequence splits. Thus, $N \in \mathrm{Add}(M)$. □

Lemma 1.17. *Let M be a left R-module. Then, $M^\perp$ is closed under epimorphisms if and only if $pd(M) \leq 1$.*

Proof. $\Rightarrow$ There exists an exact sequence $0 \to K \to R^{(X)} \to M \to 0$ for some set X. We claim that K is projective. Let N be any module. By Proposition 1.2, $\mathrm{Ext}^1(R^{(X)}, N) = 0 = \mathrm{Ext}^2(R^{(X)}, N)$. It follows that there is an exact sequence $0 \to \mathrm{Ext}^1(K, N) \to \mathrm{Ext}^2(M, N) \to 0$, and so $\mathrm{Ext}^1(K, N) \cong \mathrm{Ext}^2(M, N)$. Let $E(N)$ denote the injective hull of N and consider the exact sequence $0 \to N \to E(N) \to E(N)/N \to 0$. Again by Proposition 1.2, $\mathrm{Ext}^1(M, E(N)) = 0 = \mathrm{Ext}^2(M, E(N))$.

Therefore $\operatorname{Ext}^1(M, E(N)/N) \cong \operatorname{Ext}^2(M, N)$. Hence $\operatorname{Ext}^1(M, E(N)/N) \cong \operatorname{Ext}^1(K, N)$. Since $E(N) \in M^\perp$, $E(N)/N \in M^\perp$ by hypothesis. Thus, $\operatorname{Ext}^1(K, N) \cong \operatorname{Ext}^1(M, E(N)/N) = 0$. Since N was an arbitrary module, it follows that K is projective. Thus, $pd(M) \leq 1$.

$\Leftarrow$ Let $N \in M^\perp$ and let $\rho : N \to L$ be an epimorphism. Set $K = \operatorname{Ker}\rho$. There is an exact sequence $0 \to K \to N \xrightarrow{\rho} L \to 0$. Applying the functor $\operatorname{Hom}_R(M, \_)$ to that sequence, we get an exact sequence

$$\cdots \to \operatorname{Ext}^1(M, N) \to \operatorname{Ext}^1(M, L) \to \operatorname{Ext}^2(M, K) \to \cdots$$

It follows that $\operatorname{Ext}^2(M, K) = 0$ by Proposition 1.4 and $\operatorname{Ext}^1(M, N) = 0$ because $N \in M^\perp$. Therefore, $\operatorname{Ext}^1(M, L) = 0$. Thus, $L \in M^\perp$. □

Corollary 1.18. *Let T be a module. Then $pd(M) \leq 1$ and $\operatorname{Ext}^1(T, T^{(X)}) = 0$ for any set X if and only if $\operatorname{Gen}(T) \subseteq T^\perp$ and $T^\perp$ is closed under epimorphisms.*

Proof. By Lemma 1.17, $T^\perp$ is closed under epimorphisms if and only if $pd(T) \leq 1$. For any set X, $\operatorname{Ext}^1(T, T^{(X)}) = 0$, that is $T^{(X)} \in T^\perp$. Therefore, $\operatorname{Gen}(T) \subseteq T^\perp$. Reciprocally, if $\operatorname{Gen}(T) \subseteq T^\perp$, $\operatorname{Ext}^1(T, T^{(X)}) = 0$. □

Definition 1.19. Let M be a left R-module. The module M is called *small* if the functor $\operatorname{Hom}_R(M, \_)$ commutes with direct sums canonically.

Remark 1.20. Every finitely generated module is small

Proposition 1.21 (Lemma 1.2, [14]). *The following conditions are equivalent for a module M.*

(a) *M is small and $M^\perp$ is closed under direct sums and epimorphisms;*
(b) *M is finitely generated and $pd(M) \leq 1$.*

Proof. There exists an exact sequence

$$0 \to K \to R^{(X)} \to M \to 0 \tag{1.1}$$

for some set X. By Lemma 1.17, K is projective. It follows from [1] that $K = \bigoplus_{\alpha\in\Lambda} K_\alpha$ is a direct sum of countable generated modules K_α. Set $D = \bigoplus_{\alpha\in\Lambda} E(K_\alpha)$. There is a canonical inclusion $K \hookrightarrow D$. Since $M^\perp$ is closed under direct sums, $\operatorname{Ext}^1(M, D) = 0$. Applying $\operatorname{Hom}_R(\_, D)$ to the sequence (1.1), we get an epimorphism

$$\cdots \to \operatorname{Hom}_R(R^{(X)}, D) \to \operatorname{Hom}_R(K, D) \to 0.$$

Therefore, there exists $g : R^{(X)} \to D$ such that $g|_K = i$.

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & R^{(X)} \\ & & \downarrow{\scriptstyle i} & \swarrow{\scriptstyle g} & \\ & & D & & \end{array}$$

For each $\alpha \in \Lambda$, let $\pi_\alpha : D \to E(K_\alpha)$ and $\rho_\alpha : E(K_\alpha) \to E(K_\alpha)/K_\alpha$ be the canonical projections, respectively, and set $g_\alpha = \rho_\alpha\pi_\alpha g$, that is, the composition

$$R^{(X)} \xrightarrow{g} \bigoplus_{\alpha\in\Lambda} E(K_\alpha) \xrightarrow{\pi_\alpha} E(K_\alpha) \xrightarrow{\rho_\alpha} E(K_\alpha)/K_\alpha$$

Now, define $h : R^{(X)} \to \bigoplus_{\alpha\in\Lambda} E(K_\alpha)/K_\alpha$ as $h(x) = (g_\alpha(x))_{\alpha\in\Lambda}$. Note that $g_\alpha(K) = 0$ for each α, hence $K \leq \operatorname{Ker} h$. From (1.1), $M \cong R^{(X)}/K$. Therefore, h induces a homomorphism $\bar{h} \in \operatorname{Hom}_R\left(M, \bigoplus_{\alpha\in\Lambda} E(K_\alpha)/K_\alpha\right)$. Since M is small, there exists a finite subset $F \subseteq \Lambda$ such that $\operatorname{Im} \bar{h} \subseteq \bigoplus_{\alpha\in F} E(K_\alpha)/K_\alpha$. Thus, $\operatorname{Im} g \subseteq \bigoplus_{\alpha\in F} E(K_\alpha) + \bigoplus_{\alpha\in\Lambda} K_\alpha$. Let π denote the projection of D onto $\bigoplus_{\alpha\notin F} E(K_\alpha)$ and let $\overline{K}$ denote $\bigoplus_{\alpha\notin F} K_\alpha$. If $\bar{g} = \pi g$, then $\bar{g} \in \operatorname{Hom}_R(R^{(X)}, \overline{K})$. Since $\bar{g}|_{\overline{K}} = id$, $R^{(X)} = \operatorname{Ker}\bar{g} \oplus \overline{K}$. Set $A = \operatorname{Ker}\bar{g} \cap \overline{K} = \bigoplus_{\alpha\in F} K_\alpha$. It follows that

$$M = R^{(X)}/K = \frac{\operatorname{Ker}\bar{g} + K}{K} \cong \operatorname{Ker}\bar{g}/A.$$

Since $\operatorname{Ker}\bar{g}$ is projective, by [1], $\operatorname{Ker}\bar{g} = \bigoplus_{\beta\in\Xi}$ is a direct sum of countable generated modules. Also, A is a direct sum of countable generated modules. Hence M is direct sum of a countable generated module C and a projective module B,

$$M \cong \operatorname{Ker}\bar{g}/A = \bigoplus_{\beta\in\Xi} C_\alpha / \bigoplus_{\alpha\in F} K_\alpha \cong C \oplus B.$$

Since M is small, B is countable generated. Thus, M is a small countable generated module. Therefore M is finitely generated, by [14, Lemma 1.1].

$\Leftarrow$ By Lemma 1.17, $M^\perp$ is closed under epimorphisms. Since M is finitely generated, M is small. Now, let $\{B_i\}_{i\in I}$ be a family of modules in $M^\perp$. It is not difficult to see that, if M is small then $Ext^n(M, \_)$ commutes with direct sums for all $n > 0$. Hence,

$$\operatorname{Ext}^1\left(M, \bigoplus_{i\in I} B_i\right) \cong \bigoplus_{i\in I} \operatorname{Ext}^1(M, B_i)$$

Since each $B_i \in M^\perp$, $\operatorname{Ext}^1(M, B_i) = 0$. Therefore $\operatorname{Ext}^1\left(M, \bigoplus_{i\in I} B_i\right) = 0$. Thus, $\bigoplus_{i\in I} B_i \in M^\perp$. □

Corollary 1.22. *If M satisfies any of the conditions in Proposition 1.21, then M is finitely presented.*

Proof. Since M is finitely generated and $pd(M) \leq 1$, there is an exact sequence

$$0 \to P \to R^{(n)} \to M \to 0 \tag{1.2}$$

with P projective. Since P is projective, P is a direct summand of a free module $R^{(Y)}$. Let $j : P \to R^{(Y)}$ be the canonical inclusion, let $g : R^{(Y)} \to E(R)^{(Y)}$ be the canonical monomorphism and set $f = gj$. Since $M^\perp$ is closed under direct sums, $E(R)^{(Y)} \in M^\perp$ and so $\operatorname{Ext}^1(M, E(R)^{(Y)}) = 0$. Thus, there is an exact sequence

$$0 \to \operatorname{Hom}_R(M, E(R)^{(Y)}) \to \operatorname{Hom}_R(R^{(n)}, E(R)^{(Y)}) \to \operatorname{Hom}_R(P, E(R)^{(Y)}) \to 0.$$

This implies that f can be extended to a homomorphism $\hat{f} : R^{(n)} \to E(R)^{(Y)}$. Let $\pi_y : E(R)^{(Y)} \to E(R)$ and $\rho_y : R^{(Y)} \to R$ be the canonical projections for each $y \in Y$. Since $R^{(n)}$ is fin. gen. $F = \{y \in Y \mid \pi_y \hat{f} \neq 0\}$ is finite. It follows that ρ_y is the corestriction of $\pi_y g$ to R. Let $y \in Y \setminus F$. Then $\pi_y \hat{f} = 0$ and so $\pi_y f = 0$. This implies that $\rho_y j = \pi_y g j = \pi_y f = 0$. Hence $P = \operatorname{Im} j \subseteq R^{(F)}$ and so P is a direct summand of a fin. gen. free module. Thus, P is fin. gen. and M is finitely presented. □

Lemma 1.23. *Let M be an R-module. If M is faithful and finendo, then*

1. *there exists an exact sequence $0 \to R \xrightarrow{i} M^n \to M' \to 0$ for some $n > 0$.*
2. *for any module L, the induced homomorphism $i^* : \operatorname{Hom}_R(M, L) \to \operatorname{Hom}_R(R, L)$ is surjective if and only if $L \in \operatorname{Gen}(M)$.*
3. *$M'^\perp \subseteq \operatorname{Gen}(M)$.*

(4) *M generates every injective module.*

Proof. *(1)* Set $S = \mathrm{End}_R(M)$ and let $\{t_1, ..., t_n\}$ be a set of generators of ${}_SM$. Since M is faithful, there is a monomorphism $i : R \to M^n$ given by $i(r) = (rt_1, ..., rt_n)$. Hence, we have the exact sequence

$$0 \to R \xrightarrow{i} M^n \to M^n/R \to 0.$$

(2)$\Rightarrow$ Suppose $i^* : \mathrm{Hom}_R(M^n, L) \to \mathrm{Hom}_R(R, L)$ is surjective and let $l \in L$. Since $\mathrm{Hom}_R(R, L) \cong L$, there exists $g \in \mathrm{Hom}_R(M^n, L)$ such that $gi(1) = l$. Thus, $l \in tr^M(L)$ and so $L \in \mathrm{Gen}(M)$.

$\Leftarrow$ Applying $\mathrm{Hom}_R(\_, L)$ to the exact sequence, we get

$$0 \to \mathrm{Hom}_R(M', L) \to \mathrm{Hom}_R(M^n, L) \xrightarrow{i^*} \mathrm{Hom}_R(R, L)$$

Let $x \in L \cong \mathrm{Hom}_R(R, L)$. Since $L \in \mathrm{Gen}(M)$, there is an epimorphism $M^{(Y)} \to L$ for some set Y. Making the inverse image of x, we have a homomorphism $f : M^m \to L$ and $(x_1, ..., x_m) \in M^m$ such that $f(x_1, ..., x_m) = x$. Since ${}_SM = \langle t_1, ..., t_n\rangle$, for each $1 \leq i \leq m$ there exists $f_1^i, ..., f_n^i \in S$ such that $x_i = \sum_{j=1}^n f_j^i(t_j)$. Define $\alpha : M^n \to M^m$ as $\alpha(y_1, ..., y_n) = \left(\sum_{j=1}^n f_j^1(y_j), ..., \sum_{j=1}^n f_j^m(y_j)\right)$. Therefore, $\alpha(t_1, ..., t_n) = (x_1, ..., x_m)$. This implies that $\iota^*(f\alpha)(1) = f\alpha\iota(1) = f(x_1, ..., x_m) = x$. Proving that i^* is surjective.

(3) If $L \in M'^{\perp}$, i.e. $Ext^1(M', L) = 0$, then i^* is surjective. Thus, $M \in \mathrm{Gen}(P)$.

(4) By *(1)*, there is a monomorphism $i : R \to M^n$. Let E be an injective module and $\phi : R^{(X)} \to E$ be an epimorphism. Then ϕ can be extended to an epimorphism $\overline{\phi} : (M^n)^{(X)} \to E$. Thus, $E \in \mathrm{Gen}(M)$. □

2. Tilting Modules

Definition 2.1. A left R-module T is *tilting* if satisfies the following conditions:

(T1) There is an exact sequence $0 \to R \to T' \to T'' \to 0$ such that $T', T'' \in \mathrm{Add}(T)$.
(T2) $\mathrm{Ext}^1(T, T^{(X)}) = 0$ for any set X.
(T3) $pd(T) \leq 1$.

If, moreover, T satisfies the condition

(T4) T is finitely presented,

then T is a *classical tilting module.* A module T is a *classical partial tilting module* provided (T2),(T3) and (T4) hold true.

Proposition 2.2. *(1) A left R-module T is tilting if and only if $\mathrm{Gen}(T) = T^{\perp}$.*

(2) A left R-module P is classical partial tilting if and only if P is small, $\mathrm{Gen}(P) \subseteq P^{\perp}$ and $P^{\perp}$ is a torsion class.

(3) A left R-module T is classical tilting if and only if T is (self-) small and $\mathrm{Gen}(T) = T^{\perp}$.

Proof. *(1)* $\Rightarrow$ Suppose, T is a tilting module. Then, $\mathrm{Gen}(T) \subseteq T^{\perp}$ by Corollary 1.18. Now, by (T1), there is an exact sequence $0 \to R \xrightarrow{\alpha} T' \to T'' \to 0$ with $T', T'' \in \mathrm{Add}(T)$. Let $M \in T^{\perp}$. Then $\mathrm{Ext}^1(T'', M) = 0$ (see Proposition 1.1). Hence, there is a exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(T'', M) \longrightarrow \mathrm{Hom}_R(T', M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(R, M) \longrightarrow 0.$$

Since $\mathrm{Hom}_R(R, M) \cong M$, this implies that for each $m \in M$, there exists $g \in \mathrm{Hom}_R(T', M)$ such that $g(\alpha(1)) = m$. Therefore, M is T-generated.

$\Leftarrow$ Since $\mathrm{Gen}(T) = T^{\perp}$, $T^{\perp}$ is closed under direct sums and epimorphisms. This implies that $\mathrm{Ext}^1(T, T^{(X)}) = 0$ for any set X and $pd(M) \leq 1$ by Lemma 1.17. Since $E(R) \in \mathrm{Gen}(T)$, there is an epimorphism $\rho : T^{(X)} \to E(R)$ for some set X. Consider $i : R \to E(R)$ the canonical inclusion, then there exists a monomorphism $j : R \to T^{(X)}$ such that $\rho j = i$. This implies that T is faithful, i.e., $Ann(T) = 0$. On the other hand, since $\mathrm{Ext}^1(T, T) = 0$, by Proposition 1.1, $\mathrm{Ext}^1(T, T^Y) = 0$ for any set Y. That is $T^Y \in T^{\perp} = \mathrm{Gen}(T)$ for any set Y, thus T is finendo by Lemma 1.14. Set $S = \mathrm{End}_R(T)$ and let $\{t_1, ..., t_n\}$ be a set of generators of ${}_S T$. By Lemma 1.23.*(1)*, there is an exact sequence

$$0 \to R \overset{\iota}{\to} T^n \to T^n/R \to 0. \tag{2.1}$$

Therefore $T^n/R \in \mathrm{Gen}(T) = \mathrm{Pres}(T) = T^{\perp}$ by Lemma 1.15. This implies that there is an exact sequence

$$0 \to L \to T^{(X)} \to T^n/R \to 0 \tag{2.2}$$

with $L \in \mathrm{Gen}(T) = T^{\perp}$. Hence, applying $\mathrm{Hom}_R(\_, L)$ to the sequence (2.1), we get

$$\to \mathrm{Hom}_R(T^n, L) \overset{\iota^*}{\to} \mathrm{Hom}_R(R, L) \to \mathrm{Ext}^1(T^n/R, L) \to 0,$$

because $\mathrm{Ext}^1(T, L) = 0$. By Lemma 1.23.*(2)*, ι^* is surjective. This implies that $\mathrm{Ext}^1(T^n/R, L) = 0$. Hence, the sequence (2.2) splits by Proposition 1.8 and we get condition (T1). Thus, T is a tilting module.

(2) $\Rightarrow$ It follows from Proposition 1.21 that T is small and $P^{\perp}$ is closed under direct sums and epimorphisms. Hence $P^{\perp}$ is a torsion class. Since $P^{\perp}$ is closed under epimorphisms and by (T2), $\mathrm{Gen}(P) \subseteq P^{\perp}$.

$\Leftarrow$. By Corollary 1.22, P is finitely presented and by Proposition 1.21, $pd(P) \leq 1$. Since $\mathrm{Gen}(P) \subseteq P^{\perp}$, $Ext^1(T, T^{(X)}) = 0$ for any set X.

(3) $\Rightarrow$ By *(1)*, $\mathrm{Gen}(T) = T^{\perp}$. Since T is fin. pres., T is small.

$\Leftarrow$ By *(1)*, T is a tilting module. Since $\mathrm{Gen}(T) = T^{\perp}$, $T^{\perp}$ is a torsion class. It follows by Corollary 1.22 that T is finitely presented. $\square$

Remark 2.3. Note that the proof of *(1)* of last Proposition shows that every tilting module is faithful and finendo.

The Proposition 2.2, suggest the following generalization of classical partial tilting module.

Definition 2.4. A left R-module P is *a partial tilting module* if $\mathrm{Gen}(P) \subset P^{\perp}$ and $P^{\perp}$ is a torsion class.

Remark 2.5.
- Classical tilting module if and only if tilting and small (or just fin. gen.).
- Classical partial tilting module if and only if partial tilting and small (or just fin. gen.).
- Any direct sum of copies of a (partial) tilting module is a (partial) tilting module. (This is something which does not happen in the classical case)

A classical partial tilting module is defined as a finitely presented module satisfying (T2) and (T3). Every partial tilting module P satisfies conditions (T2) and (T3) but next example shows that conditions (T2) and (T3) are not sufficient for P to be partial tilting.

Example 2.6. Let $R=\mathbb{Z}$ and $P={}_{\mathbb{Z}}\mathbb{Q}$. Since R is a hereditary ring, $pd(P)\leq 1$. Moreover, $\operatorname{Ext}^1(P,P^{(X)})=0$ for any set X because R is Noetherian. Thus, P satisfies (T2) and (T3). Note that $\operatorname{Gen}(P)$ consists of all divisible groups. On the other hand $P^{\perp}=\{G\mid \operatorname{Ext}^1(\mathbb{Q},G)=0\}$ is the class of cotorsion goups which is not a torsion class because is not closed under direct sums. For, consider a prime number p and the abelian group $G=\bigoplus_{n>0}\mathbb{Z}_{p^n}$. It follows from [7, Corollary 54.4] that each $\mathbb{Z}_{p^n}$ is a cotorsion group but G is not. Note that $P^{\perp}$ is closed under epimorfisms (Lemma 1.17).

Clearly a summand of a classical tilting module is a classical partial tilting module. The converse is not true in general.

Example 2.7. Let k be a universal differential field of characteristic 0 with differentiation D. Denote by $R=k[y;D]$ the ring of differential polynomials of one indeterminate y over k. In [5, Theorem 1.4], it is proved that R is a left and right principal ideal domain. Hence R is Noetherian and hereditary. Moreover R has only one simple left R-module S up to isomorphism which is injective. Under this hypothesis is not difficult to prove that S is a classical partial tilting R-module. Suppose that there exists a classical tilting module T such that $S\leq^{\oplus}T$. If T is not injective, $E(T)/T$ is nonzero torsion (singular) R-module. It follows from [2, Theorem 4] that every torsion (singular) module is semisimple. Thus $E(T)/T$ is semisimple and so $E(T)/T\cong S^{(X)}$ for some set X. Therefore there exists a split monomorphism $\phi:S\to E(T)/T$. Applying $\operatorname{Hom}_R(S,\_)$ to the sequence $0\to T\to E(T)\to E(T)/T\to 0$ we get

$$\longrightarrow \operatorname{Hom}_R(S,E(T))\xrightarrow{\pi_*}\operatorname{Hom}_R(S,E(T)/T)\longrightarrow \operatorname{Ext}^1(S,T)\longrightarrow 0$$

where $\pi:E(T)\to E(T)/T$ is the canonical projection. If there exists $0\neq\psi\in\operatorname{Hom}_R(S,E(T))$ such that $\phi=\pi_*(\psi)=\pi\psi$, then $S\cong\psi(S)\leq^{\oplus}E(T)$ and $0\neq\phi(S)=\pi\psi(S)$. This implies that $\psi(S)\cap T=0$ and so $\psi(S)=0$ which is a contradiction. Thus, $\phi\notin\operatorname{Im}\pi_*$. Therefore $\operatorname{Ext}^1(S,T)\neq 0$ which cannot be because T is tilting. Hence T is injective and finitely generated. By [2, Corollary 6], T is semisimple and so $T\cong S^{(X)}$ for some set X. Therefore the condition (T1) implies that R is semisimple, contradiction. Thus, S cannot be a direct summand of a classical tilting R-module. Nevertheless, S is a direct summand of the tilting module $T=S\oplus E(R)$. By [8, Corollary 7.12], $E(R)$ is isomorphic to the skew field of fractions of R. Hence $E(R)$ is a flat non projective R-module. It follows from [11, Theorem 4.30] that $E(R)$ is not finitely generated and so T is not a classical tilting module. In fact T does not satisfies condition ($\mathrm{T1}_0$). For, suppose that there exist $n,m\in\mathbb{N}$ and an exact sequence $0\to R\overset{\nu}{\to}T^{(m)}\to T'\to 0$ with $T'\leq^{\oplus}T^{(n)}$. Since T is injective, ν can be extended to a monomorphism $\nu':E(R)\to T^{(m)}$. Hence $\nu'(E(R))/\nu(R)\leq T^{(m)}/\nu(R)\cong T'$. Thus T' contains a copy of $E(R)/R$ which is torsion and hence semisimple. Then, $E(R)/R\cong S^{(I)}$ for some infinite set I because $E(R)/R$ is not finitely generated. Therefore,

$$S^{(I)}\hookrightarrow\operatorname{Soc}(T')\leq\operatorname{Soc}(T^{(n)})=S^{(n)},$$

contradiction. Note that P is also a direct summand of $E(R)\oplus E(R)/R$ which is tilting by the next proposition.

Proposition 2.8. *Let R be a left hereditary left Noetherian ring. Then $T=E(R)\oplus E(R)/R$ is a tilting module.*

Proof. Since R is left hereditary, T is injective and $pd(T) \leq 1$. Moreover, $T^{(X)}$ is injective for any set X. Hence, $\mathrm{Ext}^1(T, T^{(X)}) = 0$ for any set X. We have an exact sequence $0 \to R \to E(R) \to E(R)/R \to 0$ with $E(R), E(R)/R \in \mathrm{Add}(T)$. Thus, T is a tilting module. □

Lemma 2.9. *Let P be a module satisfying (T2) and (T3).*

(1) Then there is a module T such that P is a summand of T, $\mathrm{Gen}(P) \subseteq P^\perp = T^\perp \subseteq \mathrm{Gen}(T)$*, and T satisfies (T1), (T3) and* $\mathrm{Ext}^1(T,T) = 0$.

(2) Let T be as in (1). Then P is partial tilting if and only if T is tilting.

Proof. *(1)* Let $\{\alpha_i\}_{i \in I}$ be a set of generators of $\mathrm{Ext}^1(P, R)$. Each α_i corresponds to an extension of R by P, that is, $0 \to R \to M_i \to P \to 0$ for all $i \in I$. Taking the direct sum over I of this sequences, we get an exact sequence $0 \to R^{(I)} \xrightarrow{j} \bigoplus_{i \in I} M_i \to P^{(I)} \to 0$. Let $h : R^{(I)} \to R$ be the homomorphism given by $h((r_i)_{i \in I}) = \sum_{i \in I} r_i$. Taking the push-out of j and h, we get a diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & M_i & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{(I)} & \xrightarrow{j} & \bigoplus_{i \in I} M_i & \longrightarrow & P^{(I)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & R & \longrightarrow & P_0 & \longrightarrow & P^{(I)} & \longrightarrow & 0 ....(\star)
\end{array}$$

This implies that $P \in \mathrm{Gen}(P_0)$. Applying the functor $\mathrm{Hom}_R(P, \_)$ to $(\star)$, we get the exact sequence

$$\longrightarrow \mathrm{Hom}_R(P, P^{(I)}) \xrightarrow{\partial} \mathrm{Ext}^1(P, R) \longrightarrow \mathrm{Ext}^1(P, P_0) \longrightarrow \mathrm{Ext}^1(P, P^{(I)}).$$

Since P satisfies (T2), $\mathrm{Ext}^1(P, P^{(I)}) = 0$. By construction ∂ is surjective (see Lemma 1.10) and hence $\mathrm{Ext}^1(P, P_0) = 0$. Thus, $P_0 \in P^\perp$. Let $M \in P^\perp$ and we apply $\mathrm{Hom}_R(\_, M)$ to $(\star)$.

$$\to \mathrm{Hom}_R(P_0, M) \xrightarrow{i^*} \mathrm{Hom}_R(R, M) \to \mathrm{Ext}^1(P^{(I)}, M) \to \mathrm{Ext}^1(P_0, M) \to \mathrm{Ext}^1(R, M)$$

Since $M \in P^\perp$, $\mathrm{Ext}^1(P^{(I)}, M) = 0$ and $\mathrm{Ext}^1(R, M) = 0$ because R is projective. Therefore, i^* is surjective and $\mathrm{Ext}^1(P_0, M) = 0$. This implies that M is P_0-generated and so $M \in P_0^\perp$. Thus, $P^\perp \subseteq \mathrm{Gen}(P_0) \cap P_0^\perp$. Put $T = P \oplus P_0$. Then $T^\perp = P^\perp \cap P_0^\perp = P^\perp$. Since $P \in \mathrm{Gen}(P_0)$, $\mathrm{Gen}(T) = \mathrm{Gen}(P_0)$. It follows that $T^\perp = P^\perp \subseteq \mathrm{Gen}(P_0) = \mathrm{Gen}(T)$. Also, note that $\mathrm{Ext}^1(P, P) = 0$ by hypothesis, $\mathrm{Ext}^1(P, P_0) = 0$ because $P_0 \in P^\perp$, $\mathrm{Ext}^1(P_0, P) = 0$ because $P^\perp \subseteq \mathrm{Gen}(P_0) \cap P_0^\perp$ and $\mathrm{Ext}^1(P_0, P_0) = 0$ because $P_0 \in P^\perp \subseteq \mathrm{Gen}(P_0) \cap P_0^\perp$. Thus, $\mathrm{Ext}^1(T, T) = 0$. By $(\star)$, T satisfies (T1). Let M be any module. Applying $\mathrm{Hom}_R(\_, M)$ to $(\star)$, for $n \geq 2$, we get

$$\to \mathrm{Ext}^{n-1}(R, M) \to \mathrm{Ext}^n(P^{(I)}, M) \to \mathrm{Ext}^n(P_0, M) \to \mathrm{Ext}^n(R, M) \to .$$

Since R is projective, $\mathrm{Ext}^n(P^{(I)}, M) \cong \mathrm{Ext}^n(P_0, M)$. Since P satisfies (T3), $\mathrm{Ext}^n(P^{(I)}, M) = 0$. This impliesthat $\mathrm{Ext}^n(P_0, M) = 0$ for all $n \geq 2$ and all module M, that is, $pd(P_0) \leq 1$. Thus, T satisfies (T3).

(2) Let T be as in *(1)*. Suppose P is partial tilting. By hypothesis, $T \in T^{\perp} = P^{\perp}$. Therefore $\text{Gen}(T) \subset T^{\perp} = P^{\perp} \subseteq \text{Gen}(T)$. By Proposition 2.2, T is a tilting module. Reciprocally, if T is tilting, $P^{\perp} = T^{\perp} = \text{Gen}(T)$ by Proposition 2.2. This implies that $P^{\perp}$ is a torsion class. By hypothesis, $\text{Gen}(P) \subseteq P^{\perp}$. Thus, P is partial tilting. □

Theorem 2.10. *Let P be a left R-module. Then, P is a partial tilting module if and only if P is a direct summand of a tilting module T such that $T^{\perp} = P^{\perp}$. Moreover, T can be chosen so that $T \cong P \oplus T$.*

Proof. $\Rightarrow$ There is a tilting module T satisfying the conditions in Lemma 2.9. Put $\overline{T} = (T \oplus P)^{(\aleph_0)}$. Since P is a direct summand of T, $\text{Gen}(T) = \text{Gen}(\overline{T})$. Also, $T^{\perp} = \overline{T}^{\perp}$ because $P^{\perp} = T^{\perp}$. Therefore $\overline{T}^{\perp} = \text{Gen}(\overline{T})$, that is, $\overline{T}$ is a tilting module. Note that $\overline{T} \cong P \oplus \overline{T}$ and $P^{\perp} = \overline{T}^{\perp}$.

$\Leftarrow$ Suppose P is a direct summand of a tilting module T with $T^{\perp} = P^{\perp}$. Then $\text{Gen}(P) \subseteq \text{Gen}(T) = T^{\perp} = P^{\perp}$. Since $P^{\perp} = T^{\perp} = \text{Gen}(T)$, $P^{\perp}$ is a torsion class. Thus, P is a partial tilting module. □

We have that, if P is partial tilting module, there exists a module C such that $T = P \oplus C$ is a tilting module and T and P have the same Ext-orthogonal class. Sometimes, C is called *the Bongartz complement of P*. Nevertheless, P can be, at the same time, a direct summand of another tilting module T' with different Ext-orthogonal class

Example 2.11. Let R be the subring of $\text{Mat}_2(\mathbb{C})$ given by $\{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \mid a \in \mathbb{R}\ b, c \in \mathbb{C}\}$. Hence R is a finite dimensional hereditary $\mathbb{R}$-algebra. Consider $P = E(R) = \text{Mat}_2(\mathbb{C})$. By Proposition 2.8, $T' = P \oplus P/R$ is a (classical) tilting module and so P is a (classical) partial tilting module. We have that

$$\mathfrak{I} = \text{Gen}(P) = \text{Gen}(T') = T'^{\perp} \subseteq P^{\perp},$$

where $\mathfrak{I}$ is the class of injective R-modules. We claim that $T'^{\perp} \neq P^{\perp}$. Put $N = \{\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) \mid a \in \mathbb{R}\ b \in \mathbb{C}\} = R\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ which is a direct summand of R. We have that ${}_RP = \{\left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right) \mid a, b \in \mathbb{C}\} \oplus \{\left(\begin{smallmatrix} 0 & d \\ 0 & c \end{smallmatrix}\right) \mid c, d \in \mathbb{C}\}$ and let P_1 and P_2 denote these direct summands respectively. Hence $P_1 = E(N)$, and $P/R = \frac{P_1 \oplus P_2}{N \oplus N'} \cong \frac{P_1}{N} \oplus \frac{P_2}{N'}$. Note that $P_1 \cong P_2$. We claim that $N \in P^{\perp} \setminus T'^{\perp}$, that is $N \in P_1^{\perp} \setminus \text{Gen}(T')$. Since N is not injective, $N \notin \text{Gen}(T')$. Write $P_1 = Rx + Ry$ with $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Let $\phi \in \text{Hom}_R(P_1, P_1/N)$ given by $\phi(x) = \left(\begin{smallmatrix} ci & 0 \\ 0 & 0 \end{smallmatrix}\right) + N$ and $\phi(y) = \left(\begin{smallmatrix} di & 0 \\ 0 & 0 \end{smallmatrix}\right) + N$ with $c, d \in \mathbb{R}$. Define $\varphi \in \text{Hom}_R(P_1, P_1)$ as $\varphi(x) = \left(\begin{smallmatrix} d+ci & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\varphi(y) = \left(\begin{smallmatrix} -c+di & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then $\phi = \pi\varphi$ where $\pi : P_1 \to P_1/N$ is the canonical projection. Hence, in the exact sequence

$$\to \text{Hom}_R(P_1, P_1) \xrightarrow{\pi_*} \text{Hom}_R(P_1, P_1/N) \to \text{Ext}^1(P_1, N) \to \text{Ext}^1(P_1, P_1)$$

π_* is surjective and $\text{Ext}^1(P_1, P_1) = 0$ because P_1 is injective. This implies that $\text{Ext}^1(P_1, N) = 0$ and so $N \in P_1^{\perp}$.

3. Tilting and classical tilting torsion theories

Definition 3.1. Let $(\mathcal{T}, \mathcal{F})$ be a (not necessarily hereditary) torsion theory in R-Mod. Then $(\mathcal{T}, \mathcal{F})$ is a *(classical) tilting torsion theory* provided there is a (classical) tilting module T such that $\mathcal{T} = T^{\perp}$. In this case, $\mathcal{T}$ is called a *(classical) tilting torsion class.*

Recall that if M is a left R-module, the *tosion theory generated by* M is the pair $(\mathcal{T}_M, \mathcal{F}_M)$ where $\mathcal{F}_M = \operatorname{Ker}\operatorname{Hom}_R(M, \_)$ and $\mathcal{T}_M = \{N \mid \operatorname{Hom}_R(N, F)\ \forall F \in \mathcal{F}_M\}$. The class $\mathcal{T}_M$ is the least torsion class containing M. It follows that $\operatorname{Gen}(M) \subset \mathcal{T}_M$ for all module M. Now, if $(\mathcal{T}, \mathcal{F})$ is a tilting torsion theory, that is, $\mathcal{T} = T^\perp$ for some tilting module T, then $\operatorname{Gen}(T) = T^\perp = \mathcal{T}$. This implies that $\mathcal{T}_T = \operatorname{Gen}(T) = \mathcal{T}$ and $\mathcal{F} = \mathcal{F}_T$. Thus, $(\mathcal{T}, \mathcal{F})$ is the torsion theory generated by T.

Theorem 3.2. *A torsion class $\mathcal{T}$ in R-Mod is a tilting torsion class if and only if $\mathcal{T} = P^\perp$ for some $P \in \mathcal{T}$.*

Proof. $\Rightarrow$ If $\mathcal{T}$ is a tilting torsion class, then $\mathcal{T} = T^\perp$ for some tilting module T. Since T is tilting, $T \in \mathcal{T}$.

$\Leftarrow$ Suppose $\mathcal{T} = P^\perp$ for some $P \in \mathcal{T}$. Then, $\operatorname{Gen}(P) \subseteq \mathcal{T} = P^\perp$. Hence P is a partial tilting module. By Theorem 2.10 there exists a tilting module T such that $T^\perp = P^\perp = \mathcal{T}$. Thus, $\mathcal{T}$ is a tilting torsion class. □

The next result shows that, if P is a (classical) partial tilting module, but not a (classical) tilting, then there are two different torsion theories determined by P.

Lemma 3.3. *If P is a partial tilting module, then both* $\operatorname{Gen}(P)$ *and $P^\perp$ are torsion classes, the second one being a tilting torsion class.*

Proof. By definition $P^\perp$ is a torsion class. It follows from Theorem 3.2 that $P^\perp$ is a tilting torsion class. It just remains to prove that $\operatorname{Gen}(P)$ is closed under extensions. Let B any module. Consider the exact sequence

$$0 \to tr^P(B) \to B \to B/tr^P(B) \to 0.$$

Applying the functor $\operatorname{Hom}_R(P, \_)$ to this sequence, we get

$$\to \operatorname{Hom}_R(P, B) \to \operatorname{Hom}_R\left(P, B/tr^P(B)\right) \to \operatorname{Ext}^1(P, tr^P(B)).$$

Since $\operatorname{Gen}(P) \subseteq P^\perp$, $\operatorname{Ext}^1(P, tr^P(B)) = 0$. This implies that for all $f \in \operatorname{Hom}_R\left(P, B/tr^P(B)\right)$ the can be lifted to a $\bar{f} \in \operatorname{Hom}_R(P, B)$. But, $\bar{f}(P) \subseteq tr^P(B)$. Hence $f = 0$. Thus, $tr^P\left(B/tr^P(B)\right) = 0$. Thus $tr^P(\_)$ is a radical and $\operatorname{Gen}(P)$ is closed under extensions. □

Proposition 3.4. *Let P be a (fin. gen.) module such that* $\operatorname{Gen}(P) \subseteq P^\perp$*. Then,* $\operatorname{Gen}(P)$ *is a (classical) tilting torsion theory if and only if P is faithful and finendo.*

Proof. $\Rightarrow$ If $\operatorname{Gen}(P) = \operatorname{Gen}(T) = T^\perp$ for some (classical) tilting module T, then P is faithful because T is faithful (Remark 2.3). Moreover, by Proposition 1.1, $P^X \in T^\perp = \operatorname{Gen}(P)$. Thus, P is finendo by Lemma 1.14.

$\Leftarrow$ Let P be a (fin. gen.) faithful and finendo module such that $\operatorname{Gen}(P) \subseteq P^\perp$. By Lemma 1.23.*(1)*, there exists an exact sequence $0 \to R \to P^n \to P' \to 0$. Let $M \in \operatorname{Gen}(P)$. By Lemma 1.23.*(3)*, $P'^\perp \subseteq \operatorname{Gen}(P) =$. On the other hand, if $M \in \operatorname{Gen}(P)$, then $\operatorname{Ext}^1(P^n, M) = 0$. Lemma 1.23.*(2)* implies that $\operatorname{Ext}^1(P', M) = 0$. Hence $M \in P'^\perp$. Put $T = P \oplus P'$. Then, $\operatorname{Gen}(P) = \operatorname{Gen}(T)$ and $T^\perp = P^\perp \cap P'^\perp = P^\perp \cap \operatorname{Gen}(P) = \operatorname{Gen}(P)$. Hence $\operatorname{Gen}(P) = \operatorname{Gen}(T) = T^\perp$ is a tilting torsion class. Note that, if P is fin. gen. then so is T. □

Definition 3.5. Let $\mathcal{T}$ be a class of modules A module P is $\mathcal{T}$-projective if the functor $\operatorname{Hom}_R(P, \_)$ preserves exactness of all sequences of the form $0 \to L \to M \to N \to 0$, where $L, M, N \in \mathcal{T}$.

Remark 3.6. If $\mathrm{Gen}(P) \subseteq P^{\perp}$, then P is $\mathrm{Gen}(P)$-projective. Indeed, let $0 \to L \to M \to N \to 0$ be an exact sequence with $L, M, N \in \mathrm{Gen}(P)$. Applying the functor $\mathrm{Hom}_R(P, \_)$ to this sequence, we get

$$0 \to \mathrm{Hom}_R(P, L) \to \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N) \to \mathrm{Ext}^1(P, L)$$

Since $\mathrm{Gen}(P) \subseteq P^{\perp}$, $\mathrm{Ext}^1(P, L) = 0$. Thus, P is $\mathrm{Gen}(P)$-projective.

Corollary 3.7. *A class of modules $\mathcal{T}$ is a (classical) tilting torsion class if and only if $\mathcal{T} = \mathrm{Gen}(P)$ for a (fin. gen.) faithful, finendo, and $\mathcal{T}$-projective module.*

Proof. $\Rightarrow$ Suppose $\mathcal{T} = \mathrm{Gen}(T) = T^{\perp}$ for some tilting module T. Then, T is faithful and finendo. By last remark, T is $\mathcal{T}$-projective.

$\Leftarrow$ By Proposition 3.4, it is enough to prove that $\mathrm{Gen}(P) \subseteq P^{\perp}$. Let $M \in \mathrm{Gen}(P)$. Consider the sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Note that $M, E(M), E(M)/M \in \mathrm{Gen}(P)$ by Lemma 1.23. Applying the functor $\mathrm{Hom}_R(P, \_)$, we get

$$\to \mathrm{Hom}_R(P, E(M)) \to \mathrm{Hom}_R(P, E(M)/M) \to \mathrm{Ext}^1(P, M) \to \mathrm{Ext}^1(P, E(M)).$$

It follows that $\mathrm{Ext}^1(P, M) = 0$ because P is $\mathrm{Gen}(P)$-projective and $\mathrm{Ext}^1(P, E(M)) = 0$. Thus, $M \in P^{\perp}$. □

Let P be a partial tilting module. Let $[\mathrm{Gen}(P), P^{\perp}]$ denote the interval of torsion classes $\mathcal{T}$ such that $\mathrm{Gen}(P) \subseteq \mathcal{T} \subseteq P^{\perp}$. The tilting torsion classes in this interval are characterized as follows.

Lemma 3.8. *Let P be a partial tilting module and let T be any module. The following conditions are equivalent:*

(a) *T is a tilting module and $P \in \mathrm{Add}(T)$;*
(b) $\mathrm{Gen}(T) = T^{\perp} \in [\mathrm{Gen}(P), P^{\perp}]$.

Proof. (a)$\Rightarrow$(b) Since T is tilting, $\mathrm{Gen}(T) = T^{\perp}$ is a torsion class. Moreover, $\mathrm{Gen}(P) \subseteq \mathrm{Gen}(T)$ and $T^{\perp} \subseteq P^{\perp}$ because $P \in \mathrm{Add}(T)$.

(b)$\Rightarrow$(a) $\mathrm{Gen}(T) = T^{\perp}$ implies that T is a tilting module. By Lemma 1.15, $P \in \mathrm{Pres}(T)$. Since $\mathrm{Gen}(T) \subseteq P^{\perp}$, $P \in \mathrm{Add}(T)$ by Lemma 1.16. □

Proposition 3.9. *Let T_1 and T_2 be two tilting modules. The following conditions are equivalent:*

(a) $T_1 \in \mathrm{Add}(T_2)$;
(b) $T_2 \in \mathrm{Add}(T_1)$;
(c) $T_1 \in T_2^{\perp}$ *and* $T_2 \in T_1^{\perp}$;
(d) $\mathrm{Gen}(T_1) = \mathrm{Gen}(T_2)$.

Proof. (a)$\Rightarrow$(d) Since T_2 is tilting and $T_1 \in \mathrm{Add}(T_2)$, by Lemma 3.8 $\mathrm{Gen}(T_1) \subseteq \mathrm{Gen}(T_2) \subseteq T_1^{\perp}$. This implies that $\mathrm{Gen}(T_1) = \mathrm{Gen}(T_2)$. (b)$\Rightarrow$(d)is similar.

(d)$\Rightarrow$(a) and (b) follows from Lemma 3.8.

(c)$\Leftrightarrow$(d) is clear because $\mathrm{Gen}(T_1) = T_1^{\perp}$ and $\mathrm{Gen}(T_2) = T_2^{\perp}$. □

Example 3.10. Let K be a field. Consider the ring of lower triangular matrices $R = \left(\begin{smallmatrix} K & 0 \\ K & K \end{smallmatrix}\right)$ with coefficients in K. We have that ${}_RR = \left(\begin{smallmatrix} K & 0 \\ K & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 0 \\ 0 & K \end{smallmatrix}\right)$. The injective hull of R is $\mathrm{Mat}_2(K) = \left(\begin{smallmatrix} K & 0 \\ K & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & K \\ 0 & K \end{smallmatrix}\right)$. There are, up to isomprphism, two simple R-modules. One is $\left(\begin{smallmatrix} 0 & 0 \\ 0 & K \end{smallmatrix}\right)$ with injective hull $P_1 = \left(\begin{smallmatrix} 0 & K \\ 0 & K \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} K & 0 \\ K & 0 \end{smallmatrix}\right)$ and the other one is $P_2 := P_1/\left(\begin{smallmatrix} 0 & 0 \\ 0 & K \end{smallmatrix}\right)$ which is injective because R is a left hereditary ring. Since R is left Artinian, every injective R-module is a direct sum of injective hulls of simple modules, that is, a direct sum of direct sums of copies of P_1 and P_2. Since P_1 generates P_2, $\mathrm{Gen}(P_1) = \mathfrak{I}$ the class of all injective modules. On the other hand, $P_1^{\perp} = R$-Mod because P_1 is projective. Hence P_1 is a partial tilting module. Now, let $M \in P_2^{\perp}$ and let N be any module. Applying $\mathrm{Hom}_R(\_, M)$ to the sequence $0 \to N \to E(N) \to E(N)/N \to 0$, we get

$$\to \mathrm{Ext}^1(E(N)/N, M) \to \mathrm{Ext}^1(E(N), M) \to \mathrm{Ext}^1(N, M) \to 0$$

We have that $E(N) = P_1^{(X)} \oplus P_2^{(Y)}$ for some sets X and Y. Since $M \in P_2^{\perp}$ and $P_1^{\perp} = R$-Mod, $\mathrm{Ext}^1(E(N).M) = 0$. Hence $\mathrm{Ext}^1(N, M) = 0$. This implies that M is injective. Since always $\mathfrak{I} \subseteq P_2^{\perp}$, then $\mathfrak{I} = P_2^{\perp}$. The class $\mathrm{Gen}(P_2)$ consist of all semisimple injective modules. Thus, P_2 is a partial tilting module.

For what follows, we will need some facts on modules of finite length. We place those results here for the convenience of the reader.

Theorem 3.11. *Let M be an indecomposable modulo of finite length. Then, $\mathrm{End}_R(M)$ is a local ringa and the noninvertible elements of $\mathrm{End}_R(M)$ are exactly the nilpotent elements.*

Theorem 3.12. *Let $M \neq 0$. If M is Artinian or Noetherian, then there exist indecomposable submodules $M_1, ..., M_n$ of M such that $M = \bigoplus_{i=1}^{n} M_i$. Moreover, if M has finite length, $\mathrm{End}_R(M_i)$ is local for every $1 \leq i \leq n$.*

Lemma 3.13. *Let M be a module of finite length. If $\mathrm{Gen}(M)$ is a torsion class, then there exists a direct summand T of M such that $\mathrm{Gen}(M) = \mathrm{Gen}(T) \subseteq T^{\perp}$.*

Proof. Since M has finite length, $M = M_1 \oplus \cdots \oplus M_n$ with M_i idecomposable. Renumbering if needed, there exists $k \leq n$ such that $M_i \in \mathrm{Gen}(M_k \oplus \cdots \oplus M_n)$ for all $1 \leq i \leq n$ and $M_i \notin \mathrm{Gen}(\bigoplus\{M_j \mid k \leq j \leq n \text{ and } i \neq j\})$ for all $k \leq i \leq n$. Put $T = M_k \oplus \cdots \oplus M_n$, then $\mathrm{Gen}(M) = \mathrm{Gen}(T)$. Let $k \leq \ell \leq n$. Suppose there is $N \in \mathrm{Gen}(M)$ with $\mathrm{Ext}^1(M_\ell, N) \neq 0$. Set $B = \bigoplus\{M_i \mid k \leq i \leq n \text{ and } i \neq \ell\}$. Then $T = M_\ell \oplus B$. Since $\mathrm{Ext}^1(M_\ell, N) \neq 0$, there is a nontrivial extension $0 \to N \to E \to M_\ell \to 0$. Hence $E \in \mathrm{Gen}(M) = \mathrm{Gen}(T)$ because $\mathrm{Gen}(M)$ is a torsion class. There is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & T^{(I)} & = & M_\ell^{(I)} \oplus B^{(I)} & & \\
 & & & & \downarrow \rho & & \downarrow ((f_i)_I, g) & & \\
0 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{\ \nu\ } & M_\ell & \longrightarrow & 0
\end{array}$$

Here $f_i = \nu\rho\eta_i$, where $\eta_i : M_\ell \to M_\ell^{(I)}$ is the canonical inclusion. Then $f_i \in \mathrm{End}_R(M_\ell)$ is not an isomorphism for all $i \in I$, because ν does not split. Since $\mathrm{End}_R(M_\ell)$ is local, $f_i \in \mathrm{Rad}(\mathrm{End}_R(M_\ell))$ for all $i \in I$. This implies that $\sum_{i \in I} f_i(M_\ell) \subseteq \mathrm{Rad}(\mathrm{End}_R(M_\ell))M_\ell$. Note that $\mathrm{Rad}(\mathrm{End}_R(M_\ell))$ is nilpotent [10, Ex. 21.24], and $M_\ell = \sum_{i \in I} f_i(M_\ell) + g\left(B^{(I)}\right)$. This implies that $M_\ell = g\left(B^{(I)}\right) \in \mathrm{Gen}(B)$ by [10, Proposition 23.16], which is a contradiction. Therefore, $\mathrm{Gen}(M) = \mathrm{Gen}(T)$ and

$$\mathrm{Gen}(M) \subseteq \bigcap_{k \leq \ell \leq n} M_\ell = T^{\perp}.$$

$\square$

The next example shows that last lemma cannot be true if the module M has infinite length.

Example 3.14. Let $p \in \mathbb{Z}$ be a prime number. Consider $R = \mathbb{Z}$ and $M = \bigoplus_{n>0} \mathbb{Z}_{p^n}$. Let $\mathcal{T}_p$ be the class of p-groups. Then $\operatorname{Gen}(M) = \mathcal{T}_p$ which is a torsion class. Let $0 \neq T \in \mathcal{T}_p$ be any p-group. It follows that $E(T) \cong \mathbb{Z}_{p^\infty}^{(X)}$ for some set X. Then,

$$\frac{\mathbb{Z}_{p^\infty}^{(X)}}{\mathbb{Z}_p^{(X)}} \cong \left(\frac{\mathbb{Z}_{p^\infty}}{\mathbb{Z}_p}\right)^{(X)} \cong \mathbb{Z}_{p^\infty}^{(X)} \cong E(T).$$

Therefore, there is a monomorphism $\alpha : T \to \frac{\mathbb{Z}_{p^\infty}^{(X)}}{\mathbb{Z}_p^{(X)}}$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p^{(X)} & \longrightarrow & A & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \mathbb{Z}_p^{(X)} & \xrightarrow{i} & \mathbb{Z}_{p^\infty}^{(X)} & \xrightarrow{\pi} & \frac{\mathbb{Z}_{p^\infty}^{(X)}}{\mathbb{Z}_p^{(X)}} & \longrightarrow & 0 \end{array}$$

where the lower row is the canonical sequence and A is the pull-back of α and π. By [13, Lemma 7.29], the upper row is exact. Since i is an essential monomorphism, the upper row is not a trivial extension. This implies that $\operatorname{Ext}^1(T, \mathbb{Z}_p^{(X)}) \neq 0$. Thus, $\operatorname{Gen}(T) \nsubseteq T^\perp$.

Theorem 3.15. *Consider the following conditions for a torsion class $\mathcal{T}$ in R-Mod.*

(1) $\mathcal{T}$ is a classic tilting torsion class.
(2) $\mathcal{T}$ is closed under direct products, it contains any injective module and $\mathcal{T} = \operatorname{Gen}(P)$ for a finitely generated module P.
(3) $\mathcal{T} = \operatorname{Gen}(P)$ for a finitely generated, faithful and finendo module P.

Then, (1)⇒(2)⇒(3). In addition, if R is left Artinian, then the three conditions are equivalent.

Proof. *(1)⇒(2)* Suppose $\mathcal{T} = \operatorname{Gen}(P) = P^\perp$ is a classical torsion class. Since $\mathcal{T} = P^\perp$, $\mathcal{T}$ is closed under direct products (Proposition 1.1) and contains any injective module. By hypothesis, P is finitely generated.

(2)⇒(3) By hypothesis, $P^X \in \operatorname{Gen}(P) = \mathcal{T}$ for every set X. This implies that P is finendo (Lemma 1.14). Since $E(R) \in \mathcal{T} = \operatorname{Gen}(P)$, there exists an epimorphism $P^{(X)} \to E(R) \to 0$ for some set X. Since R is projective, the inclusion $R \hookrightarrow E(R)$ lifts to a monomorphism $R \to P^{(X)}$. Then, P is faithful.

Now suppose R is left Artinian and assume *(3)*. Then P is of finite length. By Lemma 3.13, there is a direct summand T of P such that $\mathcal{T} = \operatorname{Gen}(P) = \operatorname{Gen}(T) \subseteq T^\perp$. Since P is faithful and there exists an epimorphism $T^{(X)} \to P$ for some set X, T is also faithful. Now, for any set X, $P^X \in \operatorname{Gen}(P)$ because P is finendo. This implies that $T^X \in \operatorname{Gen}(P) = \operatorname{Gen}(T)$. Thus, T is finendo. By Proposition 3.4, $\mathcal{T}$ is a classical tilting torsion class, proving *(1)*. □

Remark 3.16. Note that either *(1)*, *(2)* or *(3)* of Theorem 3.15 does not imply that P is of finite length. For, just consider a $P = R$ for some non left Artinian ring R.

Definition 3.17. A bimodule ${}_AC_B$ is faithfully balanced if the natural homomorphism $A \to \operatorname{End}_B(C)$ and $B \to \operatorname{End}_A(C)$ are isomorphisms.

For a nonclassical torsion class $\mathcal{T} = \mathrm{Gen}(T)$ there is not a generalization of the Brenner-Butler Theorem, that is, there is not an equivalence of categories between $\mathcal{T}$ and $\mathrm{Cogen}(\mathrm{Hom}_R(\_,T)) = \mathrm{Ker}(\mathrm{Tor}_1^S(\_,T))$. This is because T is not finitely generated. What can be done is to choose T as a classical partial tilting faithfully balanced module over its endomorphism ring such that $\mathcal{T}$ is equivalent to $\mathrm{Hom}_R(T,\mathcal{T})$.

Lemma 3.18. *Let T be an R-module with endomorphism ring $S = \mathrm{End}_R(T)$. Consider the following conditions:*

(1) T satisfies:
- (T1$_0$) *There is an exact sequence $0 \to R \to T' \to T'' \to 0$ such that $T', T'' \in \mathrm{add}(T)$.*
- (T2$_0$) $\mathrm{Ext}^1(T,T) = 0$.

(2) T is faithfully balanced as $S-R$-bimodule and ${}_ST$ is a classical partial tilting module.

(3) ${}_RT$ is faithful and there is $\bar{t} = (t_1,...,t_n) \in T^n$ such that ${}_S\langle t_1,...,t_n\rangle = {}_ST$ and $T^n/R\bar{t} \in \mathrm{add}(T)$.

(4) ${}_RT$ satisfies (T1$_0$).

Then (1)⇒(2)⇔(3)⇒(4). Moreover, if (3) is true, then every module $M \in \mathrm{Gen}(T)$ is T-reflexive, i.e., $\mathrm{Hom}_R(T,M) \otimes_S T \cong M$ canonically.

Proof. *(1)⇒(2)* Applying the functor $\mathrm{Hom}_R(\_,T)$ to the sequence (T1$_0$), we get a sequence in S-Mod:

$$0 \to \mathrm{Hom}_R(T'',T) \to \mathrm{Hom}_R(T',R) \to \mathrm{Hom}_R(R,T) \to \mathrm{Ext}_R^1(T'',T) = 0 \tag{3.1}$$

where $\mathrm{Ext}_R^1(T'',T) = 0$ because (T2$_0$). Now, we have that $T' \leq^{\oplus} T^m$ for some $m > 0$. Then $\mathrm{Hom}_R(T',T) \leq^{\oplus} \mathrm{Hom}_R(T^m,T) \cong S^m$. Therefore, $\mathrm{Hom}_R(T',T), \mathrm{Hom}_R(T'',T) \in \mathrm{add}(S)$, that is, $\mathrm{Hom}_R(T',T), \mathrm{Hom}_R(T'',T)$ are finitely generated projective S-modules. Since ${}_S\mathrm{Hom}_R(R,T) \cong {}_ST$, ${}_ST$ satisfies (T3) and (T4). Now, we apply $\mathrm{Hom}_S(\_,T)$ to (3.1) and we get the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & T' & \longrightarrow & T'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \omega_R} & & \downarrow{\scriptstyle \omega_{T'}} & & \downarrow{\scriptstyle \omega_{T''}} & & \\
0 & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(R,T),T) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(T',T),T) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(T'',T),T) & \longrightarrow & \mathrm{Ext}_S^1(T,T)
\end{array}$$

Note that $\mathrm{Hom}_S(\mathrm{Hom}_R(R,T),T) \cong \mathrm{End}_S(T)$ and $\omega_{T'}$ and $\omega_{T''}$ are isomorphisms because $T', T'' \in \mathrm{add}(T)$. Thus, ω_R is an isomorphism. This implies that T is faithfully balanced. Also, we have that $\mathrm{Ext}_S^1(T,T) = 0$.

(2)⇒(3) Since ${}_ST$ satisfies (T3) and (T4), there is an exact sequence in S-Mod

$$0 \longrightarrow K \longrightarrow S^n \xrightarrow{\ \phi\ } T \longrightarrow 0$$

with $K \in \mathrm{add}(S)$. Let $\{e_i\}$ be the canonical basis of S^n. Then, ${}_ST = {}_S\langle t_1,...,t_n\rangle$ where $t_i = \phi(e_i)$ for $1 \leq i \leq n$. Applying the functor $\mathrm{Hom}_S(\_,T)$ to the sequence, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_S(T,T) & \xrightarrow{\ \phi^*\ } & \mathrm{Hom}_S(S^n,T) & \longrightarrow & \mathrm{Hom}_S(K,T) & \longrightarrow & \mathrm{Ext}_S^1(T,T) = 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\
0 & \longrightarrow & R & \xrightarrow{\ \phi^*\ } & T^n & \longrightarrow & T^n/R\bar{t} & \longrightarrow & 0
\end{array}$$

where $\bar{t} = \phi^*(1) = (t_1, ..., t_n)$. The first isomorphism is by hypothesis and the second is the canonical isomorphism. Hence $T^n/R\bar{t} \cong \mathrm{Hom}_S(K, T) \in \mathrm{add}(T)$.

(3)⇒(2) Since ${}_RT$ is faithful and ${}_ST = {}_S\langle t_1, ..., t_n\rangle$, $R\bar{t} = R$. Hence, there is an exact sequence $0 \longrightarrow R \xrightarrow{i} T^n \longrightarrow T_0 \longrightarrow 0$, with $T_0 \cong T^n/R\bar{t}$. Applying the functor $\mathrm{Hom}_R(\_, T)$ to the sequence, we get a sequence in S-Mod:

$$0 \longrightarrow \mathrm{Hom}_R(T_0, T) \longrightarrow \mathrm{Hom}_R(T^n, T) \xrightarrow{i^*} \mathrm{Hom}_R(R, T) \cong_S T.$$

Given $t \in T$, there exist $f_1, ..., f_n \in S$ such that $t = \sum_{i=1}^n f_i(t_i)$. Then $i^*(\sum_{i=1}^n f_i)(1) = t$. Thus, i^* is surjective. Also, $T_0 \in \mathrm{add}(S)$. Therefore, ${}_ST$ satisfies (T3) and (T4). Now, we apply $\mathrm{Hom}_S(\_, T)$ and we get a commutative diagram in R-Mod,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & T^n & \longrightarrow & T_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \omega_R} & & \cong\downarrow{\scriptstyle \omega_{T^n}} & & \cong\downarrow{\scriptstyle \omega_{T_0}} & & \\
0 & \longrightarrow & \mathrm{End}_S(T) & \longrightarrow & \mathrm{Hom}_S(S^n, T) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(T_0, T), T) & \longrightarrow & \mathrm{Ext}^1(T, T),
\end{array}$$

where ω_{T^n} and ω_{T_0} are isomorphisms. Thus, ω_R is an isomoprhism. Hence, T is faithfully balanced and $\mathrm{Ext}^1_S(T, T) = 0$. At the beginning we show *(3)⇒(4)*.
For the last assertion, assume *(3)* and let $M \in \mathrm{Gen}(T)$ and $\rho_M : \mathrm{Hom}_R(T, M) \otimes_S T \to M$ be the canonical homomorphim given by $\rho_M(\phi \otimes t) = \phi(t)$. Since M is T-generated, each element $m \in M$ can be writing as a finite sum $m = \sum f_i(t_i)$ with $f_i : T \to M$. Así $\rho_M(\sum f_i \otimes t_i) = m$, that is, ρ_M is surjective. Now, let us prove that ρ_M is injective. Given any element $\sum \phi \otimes t \in \mathrm{Hom}_R(T, M) \otimes_S T$, then

$$\phi \otimes t = \phi \otimes \sum \phi_i(t_i) = \sum \phi\phi_i \otimes t_i = \sum \psi_i \otimes t_i.$$

Hence, if $u \in \mathrm{Ker}\,\rho_M$, then we can write $u = \sum_{i=1}^n \phi_i \otimes t_i$, $\phi = (\phi_1, ..., \phi_n) \in \mathrm{Hom}_R(T^n, M)$ with $\phi(\bar{t}) = 0$. Consider the following diagram:

$$\begin{array}{ccc}
T^n & \xrightarrow{\phi} & M \\
{\scriptstyle \pi}\downarrow & \overset{\bar{\phi}}{\nearrow} & \uparrow{\scriptstyle \hat{\phi}} \\
T^n/R\bar{t} & \underset{i}{\overset{\oplus}{\longrightarrow}} & T^m,
\end{array}$$

where π is the canonical projection. Since $\phi(\bar{t}) = 0$, then ϕ factors through $T^n/R\bar{t}$. Since $T^n/R\bar{t} \in \mathrm{add}(T)$, there exists $m > 0$ such that $T^n/R\bar{t} \leq^\oplus T^m$ and so there exists a homomorphism $\hat{\phi} : T^m \to M$ such that $\hat{\phi}i = \bar{\phi}$. Therefore, $\phi = \bar{\phi}\pi = \hat{\phi}i\pi = \hat{\phi}(s_{ji})$ where (s_{ji}) a matrix of $m \times n$ with $s_{ji} \in S$. Let $\eta_i : T \to T^n$ be the canonical inclusion. Hence

$$\begin{aligned} u &= \sum_{i=1}^{n} \phi_i \otimes t_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \eta_i \hat{\phi} s_{ji} \right) \otimes t_i \\ &= \sum_{j=1}^{m} \eta_j \hat{\phi} \otimes \left(\sum_{i=1}^{n} s_{ji} t_i \right) \\ &= \sum_{j=1}^{m} \eta_j \hat{\phi} \otimes (i\pi(\bar{t}))_j \\ &= 0 \end{aligned}$$

Thus, ρ_M is injective. □

The following examples show that the implications *(3)⇒(1)* and *(4)⇒(3)* are not true in general.

Example 3.19. Let K be a field.

(i) Consider the ring $R = \left(\begin{smallmatrix} K & 0 \\ K^{(\mathbb{N})} & K \end{smallmatrix}\right)$ and the idempotents $\epsilon_a = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\epsilon_b = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then,

$$R = R\epsilon_a \oplus R\epsilon_b = \begin{pmatrix} K & 0 \\ K^{(\mathbb{N})} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}.$$

Let e_1 be the first element in the canonical basis of $K^{(\mathbb{N})}$. Put $S = R\left(\begin{smallmatrix} 0 & 0 \\ e_1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ Ke_1 & 0 \end{smallmatrix}\right)$. Then $S \cong R\epsilon_b$. Set $M_1 = R\epsilon_a$, $M_2 = R\epsilon_a/S$ and $M = M_1 \oplus M_2$. Then, there is an exact sequence:

$$0 \to M_1 \oplus R\epsilon_b = R \to M_1^2 \to M \to 0.$$

This implies that M satisfies the condition *(4)* of Lemma 3.18 and M is finitely presented, since M_1 is a finitely generated projective R-module. On the other hand,

$$\operatorname{End}_R(M) = \begin{pmatrix} \operatorname{End}_R(M_1) & \operatorname{Hom}_R(M_2, M_1) \\ \operatorname{Hom}_R(M_1, M_2) & \operatorname{End}_R(M_2) \end{pmatrix} = \begin{pmatrix} K & 0 \\ K & \operatorname{End}_R(M_2) \end{pmatrix}.$$

Since the dimension over K of M_1 is infinite, M cannot be finitely generated over its endomorphism ring. Thus, M does not satisfy the condition *(3)* of Lemma 3.18.

(ii) Let R be the ring of 3×3 lower triangular matrices with coefficients in K. Then $R = R\epsilon_1 \oplus R\epsilon_2 \oplus R\epsilon_3$ where

$$\epsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \epsilon_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \epsilon_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\operatorname{Rad}(R\epsilon_1) = \left(\begin{smallmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & 0 & 0 \end{smallmatrix}\right)$ and $\operatorname{Soc}(R\epsilon_1) = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & 0 & 0 \end{smallmatrix}\right)$. Set $U = R\epsilon_1/\operatorname{Soc}(R\epsilon_1)$, $T = R\epsilon_1 \oplus R\epsilon_3 \oplus R\epsilon_1/\operatorname{Rad}(R\epsilon_1)$ and $M = T \oplus U$. Then T is a tilting module, $\dim_K(U) < \infty$ and $U \in \operatorname{Gen}(T)$. It follows from [6, Proposition 8] that M satisfies the condition *(3)* of Lemma 3.18. There is a canonical, non trivial, extension $0 \to R\epsilon_3 \to R\epsilon_1 \to U \to 0$, because $\mathrm{Soc}(R\epsilon_1) \cong R\epsilon_3$. This implies that there is a non trivial extension of T by U. Thus, $\mathrm{Ext}^1(U,T) \neq 0$ and so $\mathrm{Ext}^1(M,M) \neq 0$. That is, M does not satisfy the condition *(1)* of Lemma 3.18.

Proposition 3.20. *If T is a tilting module, then there exists a cardinal κ such that the tilting module $T^{(\kappa)}$ satisfies* ($\mathrm{T1}_0$).

Proof. By (T1), there exists an exact sequence $0 \to R \to T' \to T'' \to 0$ such that $T', T'' \in \mathrm{Add}(T)$. Then, there are two cardinals κ_1 and κ_2 such that $T' \leq^{\oplus} T^{(\kappa_1)}$ and $T'' \leq^{\oplus} T^{(\kappa_2)}$. Take $\kappa = \max\{\kappa_1, \kappa_2\}$. Thus, $T', T'' \in \mathrm{add}(T^{(\kappa)})$. □

Corollary 3.21. *Let $\mathcal{T}$ be a tilting torsion class in R-Mod. Then $\mathcal{T}$ is generated by a tilting module T such that:*

(1) if $S = \mathrm{End}_R(T)$ then T is a faithfully balanced $(S-R)$-bimodule and ${}_ST$ is a classical partial tilting module.
(2) $\mathcal{T}$ coincides with the class of T-reflexive R-modules, i.e.,

$$\mathrm{Hom}_R(T,\mathcal{T}) \underset{\mathrm{Hom}_R(T,-)}{\overset{-\otimes_S T}{\rightleftarrows}} \mathcal{T}$$

is an equivalence.
(3) $\mathrm{Hom}_R(T,\mathcal{T})$ is a torsionfree class in S-Mod if and only if ${}_RT$ is classical tilting.

Proof. Since $\mathrm{Gen}(T) = \mathrm{Gen}(T^{(\kappa)})$ for any cardinal κ, by Proposition 3.20, we can assume that $\mathcal{T} = \mathrm{Gen}(T)$ for a tilting module T satisfying ($\mathrm{T1}_0$) and ($\mathrm{T2}_0$).

(1) If follows from the condition *(2)* of Lemma 3.18.

(2) By the condition *(3)* of Lemma 3.18, every $M \in \mathrm{Gen}(T)$ is T-reflexive. On the other hand, any T reflexive module is T-generated.

(3) $\Rightarrow$ If $\mathrm{Hom}_R(T,\mathcal{T})$ is a torsionfree class in S-Mod, then $\mathrm{Hom}_R(T,\mathcal{T}) = \mathrm{Cogen}(S)$. The equivalence

$$\mathrm{Cogen}(S) \underset{\mathrm{Hom}_R(T,-)}{\overset{-\otimes_S T}{\rightleftarrows}} \mathcal{T}$$

implies that ${}_RT$ is finitely generated by [15, Theorem 1].

$\Leftarrow$ Since ${}_RT$ is finitely generated and we have the equivalence

$$\mathrm{Hom}_R(T,\mathcal{T}) \underset{\mathrm{Hom}_R(T,-)}{\overset{-\otimes_S T}{\rightleftarrows}} \mathcal{T}$$

by [12, Theorem 3.1], $\mathrm{Hom}_R(T,\mathcal{T}) = \mathrm{Cogen}(S)$. Thus, $\mathrm{Hom}_R(T,\mathcal{T})$ is a torsionfree class. □

4. Exercises

(1) Prove Corollary 1.7.
(2) [13, Ex. 7.26(ii)].
(3) Let M be a module. Prove that $M^{\perp}$ is closed under extensions and contains all injective modules.

(4) Let M be a module. Prove that $\mathrm{Gen}(M)$ is closed under epimorphisms and direct sums.
(5) Prove Remark 1.20 and give an example of a no finitely generated small module.
(6) A module T is classical tilting if and only if T satisfies ($\mathrm{T1}_0$), ($\mathrm{T2}_0$), (T3) and (T4).
(7) A module T is classical partial tilting if and only if T satisfies ($\mathrm{T2}_0$), (T3) and (T4). [9, III.6]
(8) Prove Remark 2.5.
(9) Prove that every simple module over a left hereditary left Noetherian left V-ring is a classical partial tilting module.
(10) Let R be a ring and M be a left R-module. The *singular submodule* of M is defined as $\mathcal{Z}(M) = \{m \in M \mid ann(m) \leq^{\mathrm{ess}} R\}$. It is said that a module is singular if $\mathcal{Z}(M) = M$. Show that,
   (a) $\mathcal{Z}(M/N) = M/N$ for all $N \leq^{\mathrm{ess}} M$.
   (b) if R is a semiprime Noetherian ring $\mathcal{Z}(M)$ is equal to the *torsion* of M, that is $\mathcal{Z}(M) = t(M) = \{m \in M \mid cm = 0$ for some regular element $c \in R\}$. [8, Ch. 7]
(11) Let R be a hereditary Noetherian V-ring. Using [2, Theorem 4] prove that every torsion R-module is semisimple.
(12) In Example 2.7
   (a) Prove that $T = S \oplus E(R)$ is a tilting module.
   (b) Find a reference for the sentence "$E(R)$ is a flat non projective R-module".
   (c) Prove that $E(R)/R$ cannot be finitely generated.
(13) Let E be an injective module and $\varphi : M \to E$ be a monomorphism. Show that if $\alpha : M \to N$ is an essential monomorphism, then there exists a monomorphism $\overline{\alpha} : N \to E$ such that $\overline{\alpha}\alpha = \varphi$.
(14) In Example 2.11, prove that:
   (a) R is finite dimensional over $\mathbb{R}$.
   (b) Describe the lattice of left ideals of the ring R. [10, Proposition 1.7]
   (c) Prove that R is a hereditary ring. (*Hint:* Prove that all the minimal ideals of R are isomorphic)
   (d) the left ideals $I = \left(\begin{smallmatrix} \mathbb{R} & 0 \\ \mathbb{C} & 0 \end{smallmatrix}\right)$ and $J = \left(\begin{smallmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{smallmatrix}\right)$ are two-sided ideals and are the only two maximal ideals of R. Conclude that there are only two isomorphism classes of simple R-modules
   (e) R is Artinian (use [10, Theorem 1.22])
   (f) there is an isomorphism:

$$R/J \cong P/\left(\begin{smallmatrix} \mathbb{R} & \mathbb{C} \\ \mathbb{C} & \mathbb{C} \end{smallmatrix}\right)$$

   and hence R/J is injective.
   (g) $\mathfrak{I} = \mathrm{Gen}(P)$. (*Hint:* prove that P generates the injective hull of each simple)
(15) Let r be a preradical, i.e., a subfunctor of the identity functor. Show that if r is a radical, that is, $r\left(M/r(M)\right) = 0$ for all module M, then the class $\mathcal{T}_r = \{M \mid r(M) = M\}$ is closed under extensions.
(16) in Example 3.10.
   (a) Describe the lattice of left ideals of the ring R. [10, Proposition 1.7]

(b) Prove that R is an Artinian hereditary ring. (*Hint:* Prove that all the minimal ideals of R are isomorphic)

(17) [10, Ex. 21.24].

(18) In Example 3.14, prove the equality $\mathrm{Gen}(M) = \mathcal{T}_p$.

(19) Prove that the homomorphisms $\omega_{T'}$ and $\omega_{T''}$ in the proof *(1)⇒(2)* of Lemma 3.18, are isomorphisms.

(20) Prove that the module $\mathrm{Hom}_R(M_1, M_2) = K$ and $\mathrm{Hom}_R(M_2, M_1) = 0$ in Example 3.19(i).

(21) Prove that the module T in Example 3.19(ii) is a tilting module.

(22) In the proof *(3)*⇒, prove that $\mathrm{Hom}_R(T, \mathcal{T}) = \mathrm{Cogen}(S)$.

(23) In the proof *(3)*⇐, prove that $\mathrm{Cogen}(S)$ is a torsionfree class.

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